ASYMPTOTIC WEIGHTED PERIODICITY FOR DELAY DIFFERENTIAL EQUATIONS

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ABSTRACT. Different from studying the normal asymptotic periodicity of the solution, a new kind of asymptotic periodicity—"asymptotic weighted periodicity" of the solution is established in this paper. Accordingly, a type of difference differential equation and difference differential system with time delay are discussed. Just as shown in this paper the asymptotic weighted periodic oscillations are mainly caused by the delay. Besides of the delay effect, if the impulses are taken into consideration, then the oscillations of the solution may have some kind of asymptotic weighted periodicity, and the weight is associated with the impulse.

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1. INTRODUCTION

There are so many dynamic systems related to time delays, such as the model given in [1] about the combustion control in a rocket:

$$x'(t) = (n-1)x(t) + nx(t-\tau) + u(t), \quad t > 0,$$

where n is a positive constant, $\tau > 0$ is the time delay, and $u(t)$ denotes the controlling term. At the same time, just as mentioned in [2], there are some other models of this kind:

$$x'(t) = ax(t) + bx(t-\tau) + f(t), \quad t > 0$$

and

$$x'(t) = -x(t) + A\tanh(x(t-\tau)), \quad t > 0$$

where a, b and A are fixed constants, $f(t)$ is a known function. The first one is a linear one, which describes the development of the capitalist economic crisis. The second one is a nonlinear one, which is about the dynamic system of neural network.

Mathematical speaking, for the study of this kind of difference differential equations there are many researchers, such as in [3] and the reference therein, they studied the nonlinear equation as follows:

$$x'(t) = -\alpha x(t) + f(x(t-1)), \quad t > 0,$$

where α is a positive constant and f is a piecewise constant function, which shows that this equation displays chaos and may have infinitely many periodic solutions if some conditions for α and f are satisfied.

On the other hand, the following linear equation with variable coefficients was considered in [4],

$$x'(t) = p(t)\,x(t) + q(t)\,x(t-\tau), \quad t > 0, \tag{1.1}$$

where $p(t)$, $q(t)$ are continuous functions. The results in [4] say that for $t > 0$ if

$$a(t) = -q(t)\exp\left(-\int_{t-\tau}^{t} p(s)\,ds\right) > 0, \text{ and } \liminf_{t\to\infty} \int_{t-\tau}^{t} a(s)\,ds > 1/e,$$

then all the solutions of equation (1.1) oscillate; Otherwise, if

$$0 < \limsup_{t\to\infty} \int_{t-\tau}^{t} a(s)\,ds < 3/2 \text{ and } \int_{0}^{\infty} a(s)\,ds = \infty,$$

then every solution $x(t)$ of equation (1.1) satisfies $\lim_{t\to\infty} x(t) = 0$. We think that such results are rough. For example, it is easy to check that $(1/t)\sin t$ is a solution of the following equation:

$$x'(t) = -\frac{1}{t}x(t) + \left(\frac{\pi}{2t} - 1\right) x\left(t - \frac{\pi}{2}\right), \quad t > 0.$$

Notice that $p(t) = -1/t$ and $q(t) = \pi/2t - 1$, for the case $0 < t \le \pi/2$, it is easy see $q(t) \le 0$ and hence $a(t) \le 0$. Respectively, for the case $t > \pi/2$, it's easy to calculate that $a(t) \equiv 1$ and $\int_{t-\pi/2}^{t} a(s)ds = \pi/2 > 3/2$. So the conditions given in [4] for equation (1.1) are not met, yet the solution $(1/t)\sin t$ not only oscillates but also tends to 0 as $t \to \infty$.

Respect to the periodic function and the almost periodic function, we note that $(1/t)\sin t$ is a particular function. It is well known that the periodicity phenomenon is very universal and is vastly used in the applications. Thus, it attracts numerous researchers to reveal the existence of periodic solutions for some dynamic systems. Unfortunately, not all dynamic systems have periodic solutions. Thus, the existence of almost periodic solutions become a hotspot and has been extensively investigated in recent years, such as in [5]. In addition, the existence of asymptotic periodic solutions has also been studied, such as in [6]. A function $x(t)$ is called asymptotic periodic if there exists a periodic function $\theta(t)$ such that $\lim_{t\to\infty} |x(t) - \theta(t)| = 0$. For example, the functions $(1 + 1/t)\sin t$ and $1/t + \sin t$ are asymptotic periodic respect to the periodic function $\sin t$. But for the function $(1/t)\sin t$, we see that it isn't periodic, almost periodic or asymptotic periodic in t since it oscillates about 0 and tend to 0 as $t \to \infty$. But, we think that the function $u(t) = (1/t)\sin t$ contains some kind of asymptotic periodicity, namely:

$$\lim_{t\to\infty} |u(t+2\pi) - u(t)| = \lim_{t\to\infty} |\frac{2\pi}{(t+2\pi)t} \sin t| = 0,$$

and this lead us to a new definition as follows.

Definition 1.1. *The function $x(t)$ is continuous or piecewise continuous on R. If there exists a real number $\omega > 0$ and a constant $\alpha > 0$ such that*

$$\lim_{t\to\infty} |x(t+\omega) - \alpha x(t)| = 0, \tag{1.2}$$

then we say $x(t)$ has asymptotic weighted periodicity with period ω and weight α.

Remark 1.1 The asymptotic weighted periodicity is different from the asymptotic periodicity in the normal sense: $\lim_{t\to\infty} |x(t) - \theta(t)| = 0$, where $\theta(t)$ is a periodic function with period ω. In case $\alpha = 1$, recall that $\theta(t)$ is periodic, we have $|x(t+\omega) - x(t)| \leq |x(t+\omega) - \theta(t+\omega)| + |x(t) - \theta(t)|$. Hence if $x(t)$ has asymptotic periodicity in the normal sense, then it also has asymptotic weighted periodicity with weight 1. Similarly, if $\lim_{t\to\infty} x(t) = E$ (E is a fixed number), then $|x(t+\omega) - x(t)| \leq |x(t+\omega) - E| + |x(t) - E|$, and hence $x(t)$ also has asymptotic weighted periodicity with weight 1. So the expression (1.2) include the cases $\lim_{t\to\infty} |x(t) - \theta(t)| = 0$ and $\lim_{t\to\infty} x(t) = E$ for $\alpha = 1$.

Remark 1.2 The expression (1.2) can be rewritten as: $\lim_{t\to\infty} |x(t) - \alpha x(t-\omega)| = 0$.

Correspondingly, the above arguments for $(1/t)\sin t$ implies that more general results exist for (1.1) than that in [4]. Naturally, we have some questions to ask. Are there any similar results for the nonlinear one? Are there any principles for the oscillation if the solution oscillate? To answer these questions, we consider a more general model than that in [1–4] as follows:

$$x'(t) = -p(t)\,x(t) + q(t)\,f(x(t-\tau)), \quad t > 0, \tag{1.3}$$

where $p(t)$ and $q(t)$ are continuous functions. $f(x)$ is a continuous function which satisfies the Lipschitz condition:

$$|f(x_1) - f(x_2)| \leq L|x_1 - x_2|, \quad \forall\, x_1, x_2 \in R, \tag{1.4}$$

here L is a fixed positive constant. We also assume that the associated initial function is $x_0 \in C([-\tau, 0])$, where $C([-\tau, 0])$ denotes the space of continuous functions on $[-\tau, 0]$.

Besides of the models for the combustion of rocket, the capitalist economic crisis and the neural network, the equation (1.3) can also be seen as a population protection model. Indeed, let $x(t)$ be the size of a species population in the time t. When the environments have been destroyed, the simple Malthus model is $x'(t) = -px(t)$ where p is a positive constant. Naturally, the death rate vary in time t should be more realistic. At the same time, some rare species needs protection, and our actions for protection f always lag behind and a time delay should be involved. The study of this dynamic system may also guide the artificial protection of species.

Notice that the results for (1.1) are all related to the case $q(t) < 0$, it is very natural for us to consider the case $q(t) \geq 0$ or that $q(t)$ is oscillatory. That is, in this paper we do not request that $q(t)$ has constant sign. For example, in the protection sense, we can not assure that the artificial protection is always good for species.

It's easy to see the nonlinear delay differential equation (1.3) has a smooth solution $x(t)$ for every given continuous initial function $x(t) = x_0(t)$ on $[-\tau, 0]$ by using the ordinary *Step* methods. The main aim of this paper is to reveal the asymptotic behavior of the solution.

We arrange this paper as follows: The detail arguments for problem (1.3) is given in *Section 2*. A kind of vector equation related to (1.3) is discussed in *Section 3*, and some arguments about the case when problem (1.3) is affected by impulses are given in *Section 4*. Moreover, some numerical results as applications are given in *Section 5*.

2. ASYMPTOTIC BEHAVIOR OF (1.3)

For the difference differential equation (1.3), if $f(0) = 0$, then 0 is an equilibrium. Since the coefficients $p(t), q(t)$ vary in t and may be also unbounded, it's not easy to construct a Lyapunov function to show the stability of 0, where the complexity of the nonlinear term f and the effects of the time delay should be also considered. Hence we choose another method to discuss this problem. First of all, we give some asymptotic results for the eventually nonnegative case and the eventually non-positive case.

Theorem 2.1. *Assume $p(t) > 0$ and $\int_0^\infty p(t)dt = \infty$. If $f(0) = 0$ and there exists a positive number $\mu > 1$ such that $2p(t) \geq (1+\mu)L|q(t)|$ for $t \geq 0$, and if the solution $x(t)$ of equation (1.3) is eventually nonnegative or eventually non-positive, then the solution $x(t)$ must satisfies: $\lim_{t\to\infty} x(t) = 0$.*

Proof: Assume that the solution of problem (1.3) with initial condition $x(s) = x_0(s)$ for $s \in [-\tau, 0]$ is eventually nonnegative, that is, there exists $t_1 > 0$ such that $x(t) \geq 0$ and $x(t-\tau) \geq 0$ for $t \geq t_1$. The eventually non-positive case can be discussed by the same method. If $x(t) \equiv 0$ for $t \geq t_1$, then the assertion is trivially satisfied. In the following we always assume that there exists $t^* \geq t_1$ such that $x(t^*) > 0$. Consider that $|f(x(t-\tau))| = |f(x(t-\tau)) - f(0)| \leq L|x(t-\tau) - 0| = Lx(t-\tau)$ for $t \geq t_1$, then we have

$$x'(t) = -p(t)x(t) + q(t)f(x(t-\tau)) \leq -p(t)x(t) + L|q(t)|x(t-\tau). \tag{2.1}$$

By multiplying the term $e^{\int_{t_1}^t p(s)ds}$ respect to (2.1) we get

$$[p(t)x(t)+x'(t)]e^{\int_{t_1}^t p(s)ds} \le L|q(t)|x(t-\tau)e^{\int_{t_1}^t p(s)ds},$$
$$\left[e^{\int_{t_1}^t p(s)ds}x(t)\right]' \le L|q(t)|x(t-\tau)e^{\int_{t_1}^t p(s)ds},$$
$$\int_{t_1}^t \left[e^{\int_{t_1}^w p(s)ds}x(w)\right]' dw \le \int_{t_1}^t L|q(s)|x(s-\tau)e^{\int_{t_1}^s p(\sigma)d\sigma}ds.$$

Hence for $t > t_1$ we get

$$x(t) \le e^{-\int_{t_1}^t p(s)ds}\left[x(t_1)+\int_{t_1}^t L|q(s)|x(s-\tau)e^{\int_{t_1}^s p(\sigma)d\sigma}ds\right]. \tag{2.2}$$

In the following we prove $\lim_{t\to\infty} x(t) = 0$ for several cases.

Case 1 If $x(t)$ is eventually monotonically decreasing, notice that $x(t) \ge 0$ for $t \ge t_1$, its limit must exists. Set $\lim_{t\to\infty} x(t) = E$, then $E \ge 0$. Since $\int_{t_1}^\infty p(s)ds = \infty$, from the inequality (2.2) we have

$$\begin{aligned}
E = \lim_{t\to\infty} x(t) &\le \overline{\lim_{t\to\infty}} e^{-\int_{t_1}^t p(s)ds}x(t_1) + \overline{\lim_{t\to\infty}} \frac{\int_{t_1}^t L|q(s)|x(s-\tau)e^{\int_{t_1}^s p(\sigma)d\sigma}ds}{e^{\int_{t_1}^t p(s)ds}} \\
&= 0 + \overline{\lim_{t\to\infty}} \frac{L|q(t)|x(t-\tau)e^{\int_{t_1}^t p(\sigma)d\sigma}}{p(t)e^{\int_{t_1}^t p(s)ds}} \\
&\le \frac{2}{1+\mu}\overline{\lim_{t\to\infty}} x(t-\tau) = \frac{2}{1+\mu}E, \qquad (2.3)
\end{aligned}$$

which implies $E = 0$.

Case 2 If $x(t)$ is eventually monotonically increasing, then there exists t_2, in case $t \ge t_2$, we have $0 < x(t-\tau) \le x(t)$. Notice that $\mu > 1$, from (2.1) we get

$$x'(t) \le -p(t)x(t) + L|q(t)|x(t-\tau) \le [L|q(t)| - p(t)]x(t) \le -\frac{\mu-1}{1+\mu}p(t)x(t),$$

and

$$x(t_2+\tau) \le x(t_2)\exp\left(-\int_{t_2}^{t_2+\tau}\frac{\mu-1}{1+\mu}p(s)ds\right) < x(t_2), \tag{2.4}$$

which leads to a contradiction, so this case is impossible.

Case 3 If $x(t)$ is neither eventually monotonically decreasing nor increasing. We refer to the method in [7] to continue our discussion. Of the various possibilities which then arise, we shall treat in detail the case in which $x(t)$ has an infinite sequence of local maxima $\{t_j\}$, $j = 1, 2, \cdots$ with $t_j \to \infty$ and $x(t_j) > 0$, $x'(t_j) = 0$ (here the t_1 is redefined). Other cases can be dealt with similarly. We claim that $\sup_{t\ge t_k} x(t) = x(t_k)$ for some integer k. If this is false, it means that after every maximum $x(t_j)$ there is another that is higher, so we can find a subsequence (still denoted $\{t_j\}$) such that

$x(t) < x(t_j)$ for all $t_1 \le t < t_j$. Hence from (2.1) we have

$$\begin{aligned} 0 = x'(t_j) &\le -p(t_j)x(t_j) + L|q(t_j)|x(t_j - \tau) \\ &\le -p(t_j)x(t_j) + L|q(t_j)|x(t_j) \\ &\le \left[-p(t_j) + \frac{2}{1+\mu}p(t_j)\right]x(t_j) \\ &= -\frac{\mu - 1}{1+\mu}p(t_j)x(t_j) < 0, \end{aligned} \tag{2.5}$$

which leads to a contradiction. Thus, $\sup_{t \ge t_k} x(t) = x(t_k)$ for some integer k, and we let $s_1 = t_k$. By applying this same argument to the interval $t \ge t_{k+1}$, we can infer the existence of a t_l $(l > k)$ with $\sup_{t \ge t_{k+1}} x(t) = x(t_l)$, and we let $s_2 = t_l$. This process can be continued to generate an infinite sequence $\{s_j\}$ such that $s_{j+1} > s_j$, $s_j \to \infty$, $x(t) \le x(s_j)$ for $t > s_j$ and $x'(s_j) = 0$. Furthermore, we can select a subsequence of $\{s_j\}$ such that $x(s_j - \tau) \le x(s_{j-1})$, and also denote the subsequence $\{s_j\}$. So from (2.1) for $t = s_j$,

$$\begin{aligned} 0 = x'(s_j) &\le -p(s_j)x(s_j) + L|q(s_j)|x(s_j - \tau) \\ &\le -p(s_j)x(s_j) + L|q(s_j)|x(s_{j-1}) \\ &\le \left[-x(s_j) + \frac{2}{1+\mu}x(s_{j-1})\right]p(s_j), \end{aligned} \tag{2.6}$$

which implies

$$x(s_j) \le \frac{2}{1+\mu}x(s_{j-1}).$$

Notice that $2/(1+\mu) < 1$, we have $x(s_j) \to 0$ as $j \to \infty$, which implies $x(t) \to 0$ as $t \to \infty$. The proof is complete.

Remark 2.1 Conversely, if the solution $x(t)$ of (1.3) has limit: $\lim_{t\to\infty} x(t) = E$, then the estimate $|E| \le \frac{2}{(\mu-1)L}|f(0)|$ holds true.

The reason is as follows. From (1.3) we can deduce that

$$x(t) = e^{-\int_0^t p(s)ds}\left[x(0) + \int_0^t q(s)f(x(s-\tau))e^{\int_0^s p(\sigma)d\sigma}ds\right].$$

Note that $|f(x(t-\tau))| \le |f(x(t-\tau)) - f(0)| + |f(0)| \le L|x(t-\tau)| + |f(0)|$,

$$\begin{aligned} |E| = \lim_{t\to\infty}|x(t)| &\le \overline{\lim_{t\to\infty}} e^{-\int_0^t p(s)ds}|x(0)| + \left|\overline{\lim_{t\to\infty}} \frac{\int_0^t q(s)f(x(s-\tau))e^{\int_0^s p(\sigma)d\sigma}ds}{e^{\int_0^t p(s)ds}}\right| \\ &= 0 + \left|\overline{\lim_{t\to\infty}} \frac{q(t)f(x(t-\tau))e^{\int_0^t p(\sigma)d\sigma}}{p(t)e^{\int_0^t p(s)ds}}\right| \\ &\le \frac{2}{(1+\mu)L}(L|E| + |f(0)|), \end{aligned}$$

which implies the assertion.

To demonstrate the above results, we consider a more general case of the previous model given in [2] about the dynamic system of neural network:

$$x'(t) = -p(t)\, x(t) + q(t) \tanh(x(t-\tau)), \quad t > 0,$$

where $p(t)$ and $q(t)$ are continuous functions with $p(t) > 0$ and $\int_0^\infty p(t)dt = \infty$. Since $f(x) = \tanh(x) = (e^x - e^{-x})/(e^x + e^{-x})$, we can check that the Lipschitz condition (1.4) holds for $L = 1$. If there exists a positive number $\mu > 1$ such that $2p(t) \geq (1+\mu)|q(t)|$, then its eventually nonnegative or eventually non-positive solution $x(t)$ must satisfies: $\lim_{t\to\infty} x(t) = 0$. As far as know, such fact hasn't been indicated by the previous researchers.

On the other hand, we can assume that $f(x) = -x$. In this case, we have

$$x'(t) = -p(t)\, x(t) - q(t)\, x(t-\tau), \quad t > 0.$$

Thus,

$$a(t) = q(t) \exp\left(\int_{t-\tau}^{t} p(s)\, ds\right).$$

In view of [4], we know that for the eventually positive or eventually negative solution $x(t)$, to ensure $\lim_{t\to\infty} x(t) = 0$ it needs $\int_0^\infty a(s)\, ds = \infty$. But for the case $p(t) = 2$ and $q(t) = 1/(t+1)^2$, there is no result $\lim_{t\to\infty} x(t) = 0$ since at this time $\int_0^\infty a(s)\, ds < \infty$. However, it is a direct results of *Theorem 2.1*.

Theorem 2.1 shows that for the case $f(0) = 0$ the limit of the solution for problem (1.3) must be 0 for the eventually nonnegative and the eventually non-positive cases. Yet it is not always the case, maybe the solution $x(t)$ oscillates about 0. Especially, in case $f(0) \neq 0$, the asymptotic behavior of the solution may become very complex. In the following we try to reveal this by considering the asymptotic weighted periodicity of the problem (1.3) with periodic coefficients.

Theorem 2.2. *Suppose that $p(t)$ and $q(t)$ are periodic functions with period τ/n_1 and τ/n_2 (n_1, n_2 are positive integers), respectively. If $M(t) = 2p(t) - L|q(t)| > 0$ with $\int_0^\infty M(t)dt = \infty$ and there exists a positive number $\mu > 1$ such that $2p(t) \geq (1+\mu)L|q(t)|$ for $t \geq 0$, then the solution $x(t)$ of problem (1.3) with every given initial function $x_0 \in C([-\tau, 0])$ has the following asymptotic weighted periodicity:*

$$\lim_{t\to\infty} |x(t+\tau) - x(t)| = 0. \tag{2.7}$$

Proof: Considering the periodicity of $p(t)$ and $q(t)$, from (1.3) we have

$$[x(t+\tau) - x(t)]' = -p(t)[x(t+\tau) - x(t)] + q(t)[f(x(t)) - f(x(t-\tau))]. \tag{2.8}$$

Refer to the method in [6], denotes $y(t) = x(t+\tau) - x(t)$ and multiplies (2.9) by $y(t)$,

$$\begin{aligned} \frac{1}{2}(y^2(t))' &= -p(t)y^2(t) + q(t)y(t)[f(x(t)) - f(x(t-\tau))] \\ &\leq -p(t)y^2(t) + L|q(t)| \cdot |y(t)| \cdot |y(t-\tau)| \\ &\leq \left[-p(t) + \frac{1}{2}L|q(t)|\right] y^2(t) + \frac{1}{2}L|q(t)|y^2(t-\tau), \end{aligned} \tag{2.9}$$

here the Lipschitz condition (1.4) for f is used. Furthermore, denotes $Y(t) = y^2(t)$, then $Y(t) \geq 0$ satisfies:

$$Y'(t) \leq -M(t)Y(t) + N(t)Y(t-\tau), \tag{2.10}$$

where $M(t) = 2p(t) - L|q(t)| > 0$, $N(t) = L|q(t)|$. Since $2p(t) \geq (1+\mu)L|q(t)|$, we have $M(t) \geq \mu N(t)$ with $\mu > 1$. Notice that $\int_0^\infty M(t)dt = \infty$, the same process as in *the proof* of *Theorem 2.1* reveals that $Y(t) \to 0$ as $t \to \infty$. That is, we have $\lim_{t\to\infty}[x(t+\tau) - x(t)]^2 = 0$. The proof is complete.

Remark 2.2 In *Theorem 2.2* if f satisfies $f(0) = 0$, let $x^2(t) = Y(t)$ then it is easy to prove the solution of (1.3) also satisfies $\lim_{t\to\infty} |x(t)| = 0$, where the restriction of the periodicity on $p(t), q(t)$ is not needful for this case.

Remark 2.3 For the nonhomogeneous equation

$$x'(t) = -p(t)\, x(t) + q(t)\, f(x(t-\tau)) + r(t), \quad t > 0, \tag{2.11}$$

where $r(t)$ is a continuous periodic function with period τ/n_3 (n_3 is a positive integer), it's easy to see the results in *Theorem 2.2* also hold true.

Remark 2.4 If p and q are fixed positive constants, then *Theorem 2.2* shows that the equation (1.3) has asymptotic weighted periodicity with weight 1 caused only by the time delay provided that $2p \geq (1+\mu)Lq$ with $\mu > 1$.

Consider the following equation:

$$x'(t) = -x(t) + \frac{1}{2}x(t-4\pi) + \frac{5}{2}\cos t. \tag{2.12}$$

It is a nonhomogeneous case. Accordingly, the eigenvalue problem of the homogeneous equation is

$$\lambda = -1 + \frac{1}{2}e^{-4\pi\lambda}. \tag{2.13}$$

By plotting against λ the graphs $y = \lambda + 1$ and $y = \frac{1}{2}e^{-4\pi\lambda}$, it is easy to see (2.14) has a real negative root $\lambda = -a$ with $a > 0$. It is easy to check that $x_1(t) = e^{-at} + 2\sin t + \cos t$ is a solution of (2.13) for suitable initial function on $[-4\pi, 0]$. So we have

$$\begin{aligned}
&\lim_{t\to\infty} |x_1(t+4\pi) - x_1(t)| \\
&= \lim_{t\to\infty} \left|[e^{-a(t+4\pi)} + 2\sin(t+4\pi) + \cos(t+4\pi)] - [e^{-at} + 2\sin t + \cos t]\right| \\
&= \lim_{t\to\infty} \left|e^{-a(t+4\pi)} - e^{-at}\right| \le \lim_{t\to\infty} e^{-a(t+4\pi)} + \lim_{t\to\infty} e^{-at} = 0, \qquad (2.14)
\end{aligned}$$

which accord with the results in *Theorem 2.2* for the nonhomogeneous case.

Furthermore, we see $\theta(t) = 2\sin t + \cos t$ is a periodic solution of (2.13). Refer to the *Hayes Theorem* in [7] and the results in the appendix of [8], we know all the eigenvalues of (2.14) has negative real parts, that is, $Re\lambda < 0$. Therefore, for every given initial function $x_0(t) \in C([-4\pi, 0])$, the solution should satisfies $\lim_{t\to\infty} |x(t) - (2\sin t + \cos t)| = 0$, such as $x_1(t) = e^{-at} + 2\sin t + \cos t$. This result accord with the assertion in *Remark 1.1*.

For problem (1.3), if $f(0) = 0$, then $\lim_{t\to\infty} x(t) = 0$ holds true according to *Remark 2.2*, where the restriction of periodicity on $p(t)$ and $q(t)$ isn't needful. We extend this result as follows.

Corollary 2.1. *If there exists a positive number $\mu > 1$ such that $2p(t) = (1+\mu)Lq(t) > 0$ for $t \ge 0$ with $\int_0^\infty q(t)dt = \infty$, then the solution $x(t)$ of (1.3) with every given initial function $x_0 \in C([-\tau, 0])$ has the limit:*

$$\lim_{t\to\infty} |x(t) - E| = 0, \qquad (2.15)$$

where E is the unique solution of $(1+\mu)LE = 2f(E)$ if it exists.

Proof: It is easy to see the conditions for *Theorem 2.2* are all satisfied except the request of the periodicity on $p(t)$ and $q(t)$, since $M(t) = 2p(t) - Lq(t) = \mu Lq(t) > 0$ and $\int_0^\infty M(t)dt = \int_0^\infty \mu Lq(t)dt = \mu L \int_0^\infty q(t)dt = \infty$. For the case $f(0) = 0$ the relation (2.16) is trivially satisfied. For the case $f(0) \ne 0$, notice that $2p(t) = (1+\mu)Lq(t)$, from $(1+\mu)LE = 2f(E)$ we know E is a constant solution of (1.3). So

$$[x(t) - E]' = -p(t)[x(t) - E] + q(t)[f(x(t-\tau)) - f(E)].$$

Here $p(t)$ and $q(t)$ aren't necessarily periodic in t. Let $Y(t) = (x(t) - E)^2$, then the same deduction process as in *Theorem 2.2* reveals that (2.11) also holds true with $M(t) = \mu N(t) = \mu Lq(t)$. Hence $Y(t) \to 0$ as $t \to \infty$, which implies (2.16). The proof is complete.

3. VECTOR EQUATION

Recently there has been a great interest in studying the travelling wave solutions for the system

$$x_i'(t) = p(t)\Delta^2 x_{i-1}(t) + q(t)f(x_i(t-\tau)), \qquad 1 \le i \le n$$

with or without delay, see, for example, [10-18] and the references therein. The operator Δ^2 is defined as $\Delta^2 x_{i-1}(t) = x_{i+1}(t) - 2x_i(t) + x_{i-1}(t)$ for $1 \le i \le n$. And the boundary conditions is Dirichlet type: $x_0(t) = x_{n+1}(t) = 0$ or periodic type: $x_0(t) = x_n(t), x_1(t) = x_{n+1}(t)$ for $t \ge 0$. Let $X(t)$ stand for the column vector $(x_1(t), x_2(t), \cdots, x_n(t))^T$, and $f(X(t-\tau)) = (f(x_1(t-\tau)), f(x_2(t-\tau)), \cdots, f(x_n(t-\tau)))^T$ (here the superscribe 'T' stands for the transpose), then the above system can be rewritten to be a vector equation:

$$X'(t) = p(t)AX(t) + q(t)f(X(t-\tau)),$$

where A is a $n \times n$ symmetry matrix given by:

$$A_1 = \begin{pmatrix} -2 & 1 & 0 & \cdots & 0 \\ 1 & -2 & 1 & \cdots & 0 \\ & \cdot & \cdot & \cdot & \\ 0 & \cdots & 1 & -2 & 1 \\ 0 & \cdots & 0 & 1 & -2 \end{pmatrix}_{n\times n} \quad \text{or} \quad A_2 = \begin{pmatrix} -2 & 1 & 0 & \cdots & 1 \\ 1 & -2 & 1 & \cdots & 0 \\ & \cdot & \cdot & \cdot & \\ 0 & \cdots & 1 & -2 & 1 \\ 1 & \cdots & 0 & 1 & -2 \end{pmatrix}_{n\times n}$$

respect to the Dirichlet boundary condition and the periodic boundary condition.

In this section we consider a more general vector equation as follows:

$$X'(t) = p_0(t)AX(t) - p_1(t)X(t) + q(t)f(X(t-\tau)), \tag{3.1}$$

here $X(t)$ and $f(X(t-\tau))$ are defined as above and the function f also satisfies the Lipschitz condition (1.4) (componentwise). A is a $n \times n$ real value symmetry matrix, $p_0(t), p_1(t)$ and $q(t)$ are continuous scalar functions.

We define the norm of the vector $X(t)$ as:

$$\|X(t)\| = \sqrt{\sum_{i=1}^{n} x_i^2(t)}.$$

We also say vector $X(t)$ has asymptotic weighted periodicity with period τ and weight 1 under the norm $\|\cdot\|$ if it satisfies $\lim_{t\to\infty} \|X(t+\tau) - X(t)\| = 0$. As the real value matrix A is symmetric, its eigenvalues $\lambda_1, \lambda_2, \cdots, \lambda_n$ must be real. We also assume that $\lambda_1 \ge \lambda_2 \ge \cdots \ge \lambda_n$.

Theorem 3.1. *Suppose that $p_0(t), p_1(t)$ and $q(t)$ are periodic functions with period τ/n_1, τ/n_2 and τ/n_3 (n_1, n_2, n_3 are positive integers), respectively. If $M(t) = 2[p_1(t) - \lambda_1 p_0(t)] - L|q(t)| > 0$ with $\int_0^\infty M(t)dt = \infty$ and there exists a positive number $\mu > 1$ such that $2[p_1(t) - \lambda_1 p_0(t)] \ge (1+\mu)L|q(t)|$ for $t \ge 0$, then the solution $x(t)$ of problem (3.1) with every given initial function $x_0 \in C([-\tau, 0])$ has the asymptotic weighted periodicity:*

$$\lim_{t\to\infty} \|X(t+\tau) - X(t)\| = 0. \tag{3.2}$$

Proof: Consider the periodicity of $p_0(t), p_1(t)$ and $q(t)$, from (3.1) we have

$$\begin{aligned}[X(t+\tau)-X(t)]' &= p_0(t)A[X(t+\tau)-X(t)]-p_1(t)[X(t+\tau)-X(t)] \\ &\quad +q(t)[f(X(t))-f(X(t-\tau))]. \end{aligned} \tag{3.3}$$

Denotes $Y(t)=X(t+\tau)-X(t)$ and multiplies (3.3) by the row vector $Y(t)^T$ then

$$\begin{aligned} Y(t)^TY'(t) &= p_0(t)Y(t)^TAY(t)-p_1(t)Y(t)^TY(t) \\ &\quad +q(t)Y(t)^T[f(X(t))-f(X(t-\tau))]. \end{aligned} \tag{3.4}$$

If denotes $Y(t)=(y_1(t),y_2(t),\cdots,y_n(t))^T$, then $Y(t)^TY(t)=\sum\limits_{i=1}^{n} y_i^2(t)=\|Y(t)\|^2$ and $Y(t)^TY'(t)=\sum\limits_{i=1}^{n} y_i(t)y_i'(t)=\frac{1}{2}(\|Y(t)\|^2)'$. On the other hand, the *Cauchy inequality* and the *Lipschitz condition* in (1.4) for f imply that

$$\begin{aligned} |Y(t)^T[f(X(t))-f(X(t-\tau))]| &\le \|Y(t)\|\cdot\|f(X(t))-f(X(t-\tau))\| \\ &= \|Y(t)\|\cdot\left\{\sum_{i=1}^{n}[f(x_i(t))-f(x_i(t-\tau))]^2\right\}^{1/2} \\ &\le \|Y(t)\|\cdot\left\{\sum_{i=1}^{n}L^2[x_i(t)-x_i(t-\tau)]^2\right\}^{1/2} \\ &= L\|Y(t)\|\cdot\|Y(t-\tau)\|. \end{aligned} \tag{3.5}$$

Since matrix A is symmetric, there should exists an orthogonal $n\times n$ matrix Q which satisfies $QQ^T=I$ such that

$$QAQ^T=diag(\lambda_1,\lambda_2,\cdots,\lambda_n),$$

here $diag(\lambda_1,\lambda_2,\cdots,\lambda_n)$ denotes the diagonal matrix and I is the unit matrix. Let $QY(t)=Z(t)=(z_1(t),z_2(t),\cdots,z_n(t))^T$, then

$$\begin{aligned} Y^T(t)AY(t) &= Y^T(t)Q^T\, diag(\lambda_1,\lambda_2,\cdots,\lambda_n)\, QY(t) \\ &= Z(t)^Tdiag(\lambda_1,\lambda_2,\cdots,\lambda_n)Z(t)=\sum_{i=1}^{n}\lambda_i z_i^2(t) \\ &\le \lambda_1\sum_{i=1}^{n} z_i^2(t)=\lambda_1Z(t)^TZ(t)=\lambda_1Y(t)^TQ^TQY(t) \\ &= \lambda_1Y(t)^TY(t)=\lambda_1\|Y(t)\|^2. \end{aligned} \tag{3.6}$$

From (3.4)–(3.6) it is easy to see

$$\begin{aligned} (\|Y(t)\|^2)' &\le 2[\lambda_1p_0(t)-p_1(t)]\|Y(t)\|^2+2L|q(t)|\cdot\|Y(t)\|\cdot\|Y(t-\tau)\| \\ &\le [2\lambda_1p_0(t)-2p_1(t)+L|q(t)|\,]\,\|Y(t)\|^2 \\ &\quad +L|q(t)|\cdot\|Y(t-\tau)\|^2. \end{aligned} \tag{3.7}$$

If denote $Y^*(t) = \|Y(t)\|^2$, then $Y^*(t)$ satisfies:

$$\frac{d}{dt}Y^*(t) \leq -M(t)Y^*(t) + N(t)Y^*(t-\tau), \tag{3.8}$$

where $M(t) = 2[p_1(t) - \lambda_1 p_0(t)] - L|q(t)| > 0$ and $N(t) = L|q(t)|$. Notice that $\int_0^\infty M(t)dt = \infty$, the similar arguments as that for *Theorem 2.2* reveal that $\lim_{t\to\infty} Y^*(t) = 0$, that is, $\lim_{t\to\infty} \|X(t+\tau) - X(t)\|^2 = 0$ and hence (3.2) holds true. The proof is then finished.

Remark 3.1 Relation (3.2) implies that $\lim_{t\to\infty} |x_i(t+\tau) - x_i(t)| = 0$ $(i = 1, 2, \cdots, n)$, and we can also get $\lim_{t\to\infty} |x_i(t)| = 0$ provided that $f(0) = 0$.

Example 3.1 We consider the differential-difference equation:

$$x_i'(t) = (10 + 4.5\sin 2\pi t)\Delta^2 x_{i-1}(t) + (2 + \sin 2\pi t)x_i(t-2) \tag{3.9}$$

with Dirichlet boundary conditions $x_0(t) = x_{n+1}(t) = 0$ for $t \geq 0$. Here we choose $n = 5$ for simple and hence the problem can be transferred to be the vector equation:

$$X'(t) = (10 + 4.5\sin 2\pi t)A_1 X(t) + (2 + \sin 2\pi t)X(t-2), \tag{3.10}$$

where $X(t) = (x_1(t), x_2(t), x_3(t), x_4(t), x_5(t))^T$ and A_1 is a 5×5 symmetry matrix given as the previous for Dirichlet boundary conditions. It is easy to see the eigenvalues of A_1 are $\lambda_i = -2 + 2\cos(i\pi/6)$ for $1 \leq i \leq 5$, and $\lambda_1 = -2 + 2\cos(\pi/6) = \sqrt{3} - 2$ is the biggest one.

Equation (3.10) can be seen as a particular case of equation (3.1) with $p_0(t) = 10 + 4.5\sin 2\pi t$, $p_1(t) \equiv 0$, $q(t) = 2 + \sin 2\pi t$ and $f(X(t-2)) = X(t-2)$. We check the conditions for *Theorem 3.1* as follows: It is easy to see that the period of $p_0(t), q(t)$ is $1 = \tau/2$ since $\tau = 2$. The Lipschitz constant should be $L = 1$ due to the linearity of f. So

$$\begin{aligned} M(t) &= 2[p_1(t) - \lambda_1 p_0(t)] - L|q(t)| \\ &= 2[0 - (\sqrt{3} - 2)(10 + 4.5\sin 2\pi t)] - (2 + \sin 2\pi t) \\ &= (38 - 20\sqrt{3}) + (17 - 9\sqrt{3})\sin 2\pi t \geq 38 - 20\sqrt{3} > 0, \end{aligned} \tag{3.11}$$

and it is easy to see $\int_0^\infty M(t)dt = \infty$. To satisfy $2[p_1(t) - \lambda_1 p_0(t)] \geq (1 + \mu)L|q(t)|$, only if $2[0 - (\sqrt{3} - 2)(10 + 4.5\sin 2\pi t)] \geq (1 + \mu)(2 + \sin 2\pi t)$, it suffices to show $9(2 - \sqrt{3})(2 + \sin 2\pi t) \geq (1 + \mu)(2 + \sin 2\pi t)$, i.e. $\mu \leq 17 - 9\sqrt{3} \approx 1.41154$. So μ can be chosen freely in the interval $(1, 1.41154)$. So all the conditions for *Theorem 3.1* are satisfied and the solution of (3.10) should have the asymptotic weighted periodicity: $\lim_{t\to\infty} \|X(t+2) - X(t)\| = 0$, hence the solution of the difference equation (3.9) also has the asymptotic weighted periodicity (componentwise):

$$\lim_{t\to\infty} |x_i(t+2) - x_i(t)| = 0, \qquad 1 \leq i \leq 5.$$

4. IMPULSIVE DIFFERENTIAL EQUATION

In this section we consider a differential equation with impulse effects as follows:

$$x'(t) = -p(t)x(t) + q(t)f(x(t-\tau)), \qquad t > 0, \ \ t \neq t_k, \tag{4.1}$$

$$x(t_k^+) - x(t_k^-) = bx(t_k), \tag{4.2}$$

$$x(s) = x_0(s), \quad s \in [-\tau, 0], \tag{4.3}$$

where $p(t), q(t)$ are continuous periodic functions with period τ/n_1 and τ/n_2 (n_1, n_2 are positive integers), respectively. $f(x)$ is a continuous function which satisfies the weighted Lipschitz condition: for $b \neq -1$,

$$|f(x_1) - (1+b)f(x_2)| \leq L|x_1 - (1+b)x_2| \quad \forall\, x_1, x_2 \in R, \tag{4.4}$$

here L is a fixed positive constant. $\{t_k\}_0^\infty$ are discrete points with $\lim\limits_{k\to\infty} t_k = \infty$. $x(t_k^+) = \lim\limits_{t\to t_k+0} x(t)$ and $x(t_k^-) = \lim\limits_{t\to t_k-0} x(t)$ with $x(t_k) = x(t_k^-)$. Relation (4.2) is the impulsive condition which describes the instantaneous changes in the substance x studied, and constant b is the jump. The initial function $x_0(s)$ is a continuous function on $[-\tau, 0]$. For the study of the periodicity of the impulsive differential equations there are many researchers, such as in [19–21], yet the methods they have used aren't suitable for solving our problem though it's only a relatively simpler ordinary differential equation due to the variation of the coefficients and the effects of the delay and impulses. Here we prefer showing the asymptotic weighted periodicity of the solution for initial value problem (4.1)–(4.3) rather than discussing the existence of the periodic solution for (4.1)(4.2). As shown in *Section 2*, the delay in the problem (4.1)(4.3) with no impulse can cause the solution vary weighted periodically with weight 1 as $t \to \infty$, but if the impulse is taken into consideration, as shown by the following results, the oscillation of the solution may have some kind of asymptotic weighted periodicity with some fixed weight related to the jump b.

In the following we take $t_k = k\tau$ ($k = 0, 1, \cdots.$) for simple to continue our discussion.

Notice that $x(t_k^-) = x(t_k)$, relation (4.2) can be rewritten as $x(t_k^+) = (1+b)x(t_k)$. Denote $x_1(t)$ the solution of the problem (4.1)–(4.3) on $(0, \tau]$. In this case, $-\tau < t - \tau \leq 0$, and $x_1(t-\tau) = x_0(t-\tau)$. If we extend x_0 by $x_0(t) = x_0(t-\tau)$ on $(0, \tau]$, then $x_1(t-\tau) = x_0(t)$ and x_1 solves

$$\begin{cases} x_1'(t) = -p(t)x_1(t) + q(t)f(x_0(t)), & t \in (0, \tau], \\ x_1(0) = (1+b)x_0(0), \end{cases} \tag{4.5}$$

where $x_1(0)$ stands for $x_1(0^+)$. As $x_0(t)$ is a known function, the linear problem (4.5) has a unique smooth solution $x_1(t)$. We can also extend x_1 by $x_1(t) = x_1(t-\tau)$ on

$(\tau, 2\tau]$ and get

$$\begin{cases} x_2'(t) = -p(t)x_2(t) + q(t)f(x_1(t)), & t \in (\tau, 2\tau], \\ x_2(\tau) = (1+b)x_1(\tau). \end{cases} \tag{4.6}$$

According to this method, for every integer $k = 1, 2, \cdots,$

$$\begin{cases} x_k'(t) = -p(t)x_k(t) + q(t)f(x_{k-1}(t)), & t \in ((k-1)\tau, k\tau], \\ x_k((k-1)\tau) = (1+b)x_{k-1}((k-1)\tau), \end{cases} \tag{4.7}$$

where $x_k((k-1)\tau)$ stands for $x_k\left(((k-1)\tau)^+\right)$ and x_{k-1} satisfy the extension relations $x_{k-1}(t) = x_{k-1}(t-\tau)$ on $((k-1)\tau, k\tau]$, and further periodic extensions for x_{k-1} are permitted if it needs. If we have solved x_{k-1}, then we can take it as a known function and continue to solve x_k.

The iteration process (4.7) implies the following relations for $t \in (k\tau, (k+1)\tau]$:

$$\begin{cases} [x_{k+1} - (1+b)x_k]' = -p(t)[x_{k+1} - (1+b)x_k] + q(t)[f(x_k) - (1+b)f(x_{k-1})], \\ [x_{k+1} - (1+b)x_k]|_{t=k\tau} = (1+b)[x_k(k\tau) - (1+b)x_{k-1}((k-1)\tau)]. \end{cases} \tag{4.8}$$

As x_0 is continuous on $[-\tau, 0]$ it should be bounded and there exists a positive constant K such that $-K \le x_0 \le K$. Similarly, as x_1 is a smooth solution of problem (4.5), we can set $C_1 = \inf\{x_1(t);\ t \in (0,\tau]\}$ and $C_2 = \sup\{x_1(t);\ t \in (0,\tau]\}$. We also denote:

$$\begin{aligned} & C = \max\{|C_1 - (1+b)K|^2,\ |C_2 + (1+b)K|^2\}; \\ & M(t) = 2p(t) - L|q(t)|; \qquad N(t) = L|q(t)|; \\ & S = [(1+b)^2 - 1/\mu] \cdot \exp\left(-\int_0^{\tau/2} M(s)ds\right) + 1/\mu; \\ & T = [(1+b)^2 - 1/\mu] \cdot \exp\left(-\int_0^{\tau} M(s)ds\right) + 1/\mu; \\ & P = \max\{(1+b)^2,\ S\}; \quad Q = \begin{cases} (1+b)^2 S, & (1+b)^2 > 1/\mu, \\ T, & (1+b)^2 \le 1/\mu, \end{cases} \end{aligned} \tag{4.9}$$

here μ is a positive constant to be defined.

Theorem 4.1. *In case $b > -1$, if we can choose a positive number $\mu > 1$ such that $2p(t) \ge (1+\mu)L|q(t)|$ for $t \ge 0$ and $Q < 1$, then the solution $x(t)$ of problem (4.1)–(4.3) has the following asymptotic weighted periodicity:*

$$\lim_{t\to\infty} |x(t) - (1+b)x(t-\tau)| = 0. \tag{4.10}$$

Proof: For every $t \in ((k-1)\tau, k\tau]$, let $y_k(t) = x_k(t) - (1+b)x_{k-1}(t)$ for $k = 1, 2, \cdots,$ we can multiply the first equation in (4.8) by $y_{k+1}(t)$ and get

$$y_{k+1}(t)y_{k+1}'(t) = -p(t)y_{k+1}^2(t) + q(t)y_{k+1}(t)[f(x_k(t)) - (1+b)f(x_{k-1}(t))]. \tag{4.11}$$

Consider that $b > -1$, the weighted Lipschitz condition (4.4) for f implies that: $|f(x_k(t)) - (1+b)f(x_{k-1}(t))| \leq L|x_k(t) - (1+b)x_{k-1}(t)| = L|y_k|$, so from (4.11) we have

$$\begin{aligned} \frac{1}{2}(y_{k+1}^2(t))' &\leq -p(t)y_{k+1}^2(t) + L|q(t)| \cdot |y_{k+1}(t)| \cdot |y_k(t)| \\ &\leq \left[-p(t) + \frac{1}{2}L|q(t)|\right] y_{k+1}^2(t) + \frac{1}{2}L|q(t)|y_k^2(t). \end{aligned} \tag{4.12}$$

Denotes $Y_k(t) \equiv y_k^2(t)$, also use the denotations in (4.9), (4.12) can be rewritten as:

$$Y_{k+1}'(t) \leq -M(t)Y_{k+1}(t) + N(t)Y_k(t), \quad \forall\, t \in (k\tau, (k+1)\tau].$$

Furthermore, considering the extension property of x_k, from (4.8) it is easy to see $Y_{k+1}(k\tau) = (1+b)^2 Y_k(k\tau)$. Condition $2p(t) \geq (1+\mu)L|q(t)|$ implies that $M(t) \geq \mu N(t) \geq 0$. So for $t \in (k\tau, (k+1)\tau]$ from (4.13) we can obtain

$$\begin{aligned} Y_{k+1}(t) &\leq e^{-\int_{k\tau}^t M(s)ds}\left[Y_{k+1}(k\tau) + \int_{k\tau}^t N(s)Y_k(s)e^{\int_{k\tau}^s M(\sigma)d\sigma}ds\right] \\ &\leq e^{-\int_{k\tau}^t M(s)ds}\left[(1+b)^2 Y_k(k\tau) + \int_{k\tau}^t Y_k(s)\frac{1}{\mu}M(s)e^{\int_{k\tau}^s M(\sigma)d\sigma}ds\right] \\ &= e^{-\int_{k\tau}^t M(s)ds}\left[(1+b)^2 Y_k(k\tau) + \frac{1}{\mu}\int_{k\tau}^t Y_k(s)\left(e^{\int_{k\tau}^s M(\sigma)d\sigma}\right)' ds\right]. \end{aligned} \tag{4.13}$$

As $b > -1$ and so $1 + b > 0$, from the denotations in (4.9) for $t \in (0, \tau]$,

$$\begin{aligned} Y_1(t) &= |x_1(t) - (1+b)x_0(t)|^2 \\ &\leq \max\{|C_1 - (1+b)K|^2,\ |C_2 + (1+b)K|^2\} \equiv C. \end{aligned} \tag{4.14}$$

(4.14) and (4.15) imply that

$$\begin{aligned} Y_2(t) &\leq e^{-\int_\tau^t M(s)ds}\left[(1+b)^2 Y_1(\tau) + \frac{1}{\mu}\int_\tau^t Y_1(s)\left(e^{\int_\tau^s M(\sigma)d\sigma}\right)' ds\right] \\ &\leq e^{-\int_\tau^t M(s)ds}\left[(1+b)^2 C + \frac{1}{\mu}\int_\tau^t C\left(e^{\int_\tau^s M(\sigma)d\sigma}\right)' ds\right] \\ &= C\left\{\left[(1+b)^2 - \frac{1}{\mu}\right] e^{-\int_\tau^t M(s)ds} + \frac{1}{\mu}\right\}, \quad t \in (\tau, 2\tau]. \end{aligned} \tag{4.15}$$

Case 1 If $(1+b)^2 > 1/\mu$, then $M(t) \geq 0$ implies that:

$$\begin{aligned} Y_2(t) &\leq C\left\{\left[(1+b)^2 - \frac{1}{\mu}\right] e^0 + \frac{1}{\mu}\right\} = C(1+b)^2, \quad t \in (\tau, 3\tau/2], \\ Y_2(t) &\leq C\left\{\left[(1+b)^2 - \frac{1}{\mu}\right] e^{-\int_\tau^{3\tau/2} M(s)ds} + \frac{1}{\mu}\right\} \\ &= C\left\{\left[(1+b)^2 - \frac{1}{\mu}\right] e^{-\int_0^{\tau/2} M(s)ds} + \frac{1}{\mu}\right\} \equiv CS,\ t \in (3\tau/2, 2\tau], \end{aligned} \tag{4.16}$$

here the periodicity of $M(t)$ is used. It is easy to see $(1+b)^2 \geq S$ and then

$$\begin{aligned} Y_3(t) &\leq e^{-\int_{2\tau}^{t} M(s)ds} \left[(1+b)^2 Y_2(2\tau) + \frac{1}{\mu}\int_{2\tau}^{t} Y_2(s)\left(e^{\int_{2\tau}^{s} M(\sigma)d\sigma}\right)' ds\right] \\ &\leq e^{-\int_{2\tau}^{t} M(s)ds} \left[(1+b)^2 CS + \frac{1}{\mu}\int_{2\tau}^{t} C(1+b)^2\left(e^{\int_{2\tau}^{s} M(\sigma)d\sigma}\right)' ds\right] \\ &= C(1+b)^2\left\{\left[(1+b)^2 - \frac{1}{\mu}\right] e^{-\int_0^{\tau/2} M(s)ds - \int_{2\tau}^{t} M(s)ds} + \frac{1}{\mu}\right\} \\ &\leq C(1+b)^2 S, \qquad t \in (2\tau, 3\tau] \end{aligned} \tag{4.17}$$

and

$$\begin{aligned} Y_4(t) &\leq e^{-\int_{3\tau}^{t} M(s)ds} \left[(1+b)^2 Y_3(3\tau) + \frac{1}{\mu}\int_{3\tau}^{t} Y_3(s)\left(e^{\int_{3\tau}^{s} M(\sigma)d\sigma}\right)' ds\right] \\ &\leq e^{-\int_{3\tau}^{t} M(s)ds} \left[(1+b)^2 C(1+b)^2 S + \frac{1}{\mu}\int_{3\tau}^{t} C(1+b)^2 S\left(e^{\int_{3\tau}^{s} M(\sigma)d\sigma}\right)' ds\right] \\ &= C(1+b)^2 S\left\{\left[(1+b)^2 - \frac{1}{\mu}\right] e^{-\int_{3\tau}^{t} M(s)ds} + \frac{1}{\mu}\right\}, \qquad t \in (3\tau, 4\tau] \end{aligned}$$

which implies that

$$Y_4(t) \leq C(1+b)^2 S\left\{\left[(1+b)^2 - \frac{1}{\mu}\right] e^0 + \frac{1}{\mu}\right\} = C(1+b)^4 S, \quad t \in (3\tau, 7\tau/2],$$

$$Y_4(t) \leq C(1+b)^2 S^2, \quad t \in (7\tau/2, 4\tau]. \tag{4.18}$$

From the iteration process we can get the estimate for every $m = 1, 2, \cdots$ as:

$$\begin{aligned} &Y_{2m}(t) \leq C(1+b)^2[(1+b)^2 S]^{m-1}, \qquad t \in ((2m-1)\tau, (2m-1/2)\tau], \\ &Y_{2m}(t) \leq CS[(1+b)^2 S]^{m-1}, \qquad t \in ((2m-1/2)\tau, 2m\tau], \\ &Y_{2m+1}(t) \leq C[(1+b)^2 S]^m, \qquad t \in (2m\tau, (2m+1)\tau]. \end{aligned} \tag{4.19}$$

Consider the denotations $P = \max\{(1+b)^2,\ S\}$ and $Q = (1+b)^2 S$ for case $(1+b)^2 > 1/\mu$, it's easy to see

$$\begin{aligned} &Y_{2m}(t) \leq CPQ^{m-1}, \qquad t \in ((2m-1)\tau, 2m\tau], \\ &Y_{2m+1}(t) \leq CQ^m, \qquad t \in (2m\tau, (2m+1)\tau]. \end{aligned} \tag{4.20}$$

For each $t > 0$, there exists a nonnegative integer k and $t_0 \in (0, \tau]$ such that $t = k\tau + t_0$, here $k = 2m-1$ or $2m$. Considering that $Q < 1$,

$$\begin{aligned} &\overline{\lim_{m\to\infty}} Y_{2m}((2m-1)\tau + t_0) \leq \lim_{m\to\infty} CPQ^{m-1} = 0, \quad \forall\, t_0 \in (0, \tau], \\ &\overline{\lim_{m\to\infty}} Y_{2m+1}(2m\tau + t_0) \leq \lim_{m\to\infty} CQ^m = 0, \ \forall\, t_0 \in (0, \tau]. \end{aligned} \tag{4.21}$$

If let $k \to \infty$ then for every $t_0 \in (0, \tau]$,

$$\overline{\lim_{k\to\infty}} |x(k\tau + t_0) - (1+b)x((k-1)\tau + t_0)|^2 = \overline{\lim_{k\to\infty}} Y_{k+1}(k\tau + t_0) = 0.$$

The arbitrariness of $t_0 \in (0,\tau]$ implies $\overline{\lim_{t\to\infty}}|x(t)-(1+b)x(t-\tau)|^2 = 0$, and hence relation (4.10) holds true for the case $(1+b)^2 > 1/\mu$.

Case 2 If $(1+b)^2 \le 1/\mu$, then $M(t) \ge 0$ implies that:

$$\begin{aligned} Y_2(t) &\le C\left\{\left[(1+b)^2 - \frac{1}{\mu}\right] e^{-\int_\tau^{2\tau} M(s)ds} + \frac{1}{\mu}\right\} \\ &= C\left\{\left[(1+b)^2 - \frac{1}{\mu}\right] e^{-\int_0^{\tau} M(s)ds} + \frac{1}{\mu}\right\} \equiv CT, \quad t \in (\tau, 2\tau]. \end{aligned} \tag{4.22}$$

$$\begin{aligned} Y_3(t) &\le e^{-\int_{2\tau}^t M(s)ds}\left[(1+b)^2 Y_2(2\tau) + \frac{1}{\mu}\int_{2\tau}^t Y_2(s)\left(e^{\int_{2\tau}^s M(\sigma)d\sigma}\right)' ds\right] \\ &\le e^{-\int_{2\tau}^t M(s)ds}\left[(1+b)^2 CT + \frac{1}{\mu}\int_{2\tau}^t CT\left(e^{\int_{2\tau}^s M(\sigma)d\sigma}\right)' ds\right] \\ &= CT\left\{\left[(1+b)^2 - \frac{1}{\mu}\right] e^{-\int_{2\tau}^t M(s)ds} + \frac{1}{\mu}\right\} \le CT^2, \ \ t \in (2\tau, 3\tau]. \end{aligned} \tag{4.23}$$

From the iteration process we can get the estimate for every $k = 1, 2, \cdots$ as:

$$Y_{k+1}(t) \le CT^k = CQ^k, \qquad t \in (k\tau, (k+1)\tau].$$

then for every $t_0 \in (0,\tau]$,

$$\begin{aligned} &\overline{\lim_{k\to\infty}}|x(k\tau+t_0) - (1+b)x((k-1)\tau+t_0)|^2 \\ &= \overline{\lim_{k\to\infty}} Y_{k+1}(k\tau+t_0) \le \lim_{k\to\infty} CQ^k = 0 \end{aligned} \tag{4.24}$$

due to $Q < 1$, and hence $\overline{\lim_{t\to\infty}}|x(t)-(1+b)x(t-\tau)|^2 = 0$, which implies relation (4.10). The proof is then proven.

If C in (4.9) is substituted by $C^* = \max\{|C_2 - (1+b)K|^2,\ |C_1 + (1+b)K|^2\}$, then we can prove that *Theorem 4.1* also holds true for $b < -1$ by using the same method. According to *Definition 1.1* the weight should be positive, so we can change the form of the asymptotic weighted periodicity. Since

$$\begin{aligned} &\lim_{t\to\infty}\left|x(t) - (1+b)^2 x(t-2\tau)\right| \\ &= \lim_{t\to\infty}\left|x(t) - (1+b)x(t-\tau) + (1+b)x(t-\tau) - (1+b)^2 x(t-2\tau)\right| \\ &\le \lim_{t\to\infty}|x(t)-(1+b)x(t-\tau)| + |1+b| \cdot \lim_{t\to\infty}|x(t-\tau)-(1+b)x(t-2\tau)| \\ &= 0 + |1+b|\cdot 0 = 0, \end{aligned} \tag{4.25}$$

we can get a result for the case $b < -1$ as follows.

Theorem 4.2. *In case $b < -1$, if we can choose a positive number $\mu > 1$ such that $2p(t) \ge (1+\mu)L|q(t)|$ for $t \ge 0$ and $Q < 1$ with Q defined in (4.9), then the solution $x(t)$ of problem (4.1)–(4.3) has the following asymptotic weighted periodicity:*

$$\lim_{t\to\infty}\left|x(t) - (1+b)^2 x(t-2\tau)\right| = 0.$$

Remark 4.1 In *Theorem 4.1* and *Theorem 4.2*, for the case $(1+b)^2 > 1/\mu$, since $Q = (1+b)^2 S > (1+b)^2/\mu$, to satisfy $Q < 1$ it needs $1/\mu < (1+b)^2 < \mu$; for the case $(1+b)^2 \le 1/\mu$, it is easy to see $Q = T \le 1/\mu < 1$ is trivially satisfied.

Remark 4.2 If $b = 0$ then there isn't impulse and the results in *Theorem 4.1* accord with that in *Theorem 2.2*, the difference is that there haven't the restriction $M(t) = 2p(t) - L|q(t)| > 0$ and $\int_0^\infty M(t) = \infty$. If $b = -2$, then *Theorem 4.2* also implies that the solution has asymptotic weighed periodicity with period 2τ and weight 1, which also includes the case $\lim_{t\to\infty} |x(t)| = 0$ for $f(0) = 0$ due to $(1+b)^2 = 1$ at this time.

5. SIMULATIONS

In this section, we give some numerical results for the asymptotic behavior of the solution to the impulsive equation (4.1)–(4.3).

Example 5.1 For simplicity of simulations we consider the linear problem as follows:

$$\begin{cases} x'(t) = -p(t)x(t) + q(t)f(x(t-\tau)), & t > 0, \ \ t \neq t_k, \\ x(t_k^+) = (1+b)x(t_k), & \\ x(s) = x_0(s), \quad s \in [-\tau, 0], & \end{cases}$$

where $f(x(t-\tau)) = x(t-\tau) + C$ (C is a fixed constant), $t_k = k\tau$ for $k = 1, 2, \cdots$, $p(t) = 1.5\sin 2\pi t$, $q(t) = \sin 2\pi t$ with period 1 and $x_0(s) = s + 3$.

Case 1 Choose $b = -0.2$, $\tau = 2$ and $C = 0$. We check the conditions for *Theorem 4.1* as follows: Since $f(x(t-\tau)) = x(t-\tau)$, it is easy to see $f(0) = 0$ and $L = 1$. If we choose $\mu = 4/3$, then $2p(t) = 3\sin 2\pi t \ge (1+4/3)\sin 2\pi t = (1+\mu)L|q(t)|$. At the same time, from the denotations in (4.9), $M(t) = 2p(t) - Lq(t) = 2 \times 1.5\sin 2\pi t - \sin 2\pi t = 2\sin 2\pi t$ and $(1+b)^2 = (1-0.2)^2 = 0.64 < 3/4 = 1/\mu$. So

$$\begin{aligned} Q = T &= \left[(1+b)^2 - 1/\mu\right] \cdot \exp\left(-\int_0^\tau M(s)ds\right) + 1/\mu \\ &= [0.64 - 3/4] \cdot \exp\left(-\int_0^2 2\sin(2\pi s)ds\right) + 3/4 < 1, \end{aligned}$$

hence all the conditions for *Theorem 4.1* are satisfied. So the solution $x(t)$ of problem (5.1) has the following asymptotic weighted periodicity:

$$\lim_{t\to\infty} |x(t) - 0.8x(t-2)| = 0.$$

Since the amplitude of $x(t)$ may get nearly to be 0.8 times after 2 units of time t, it's easy to see $x(t) \to 0$ as $t \to \infty$ just as shown in *Figure 1*.

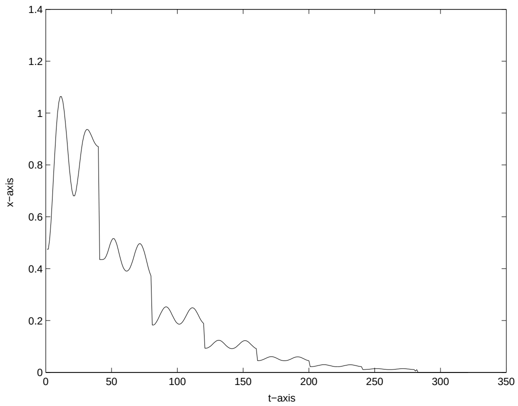

Figure 1

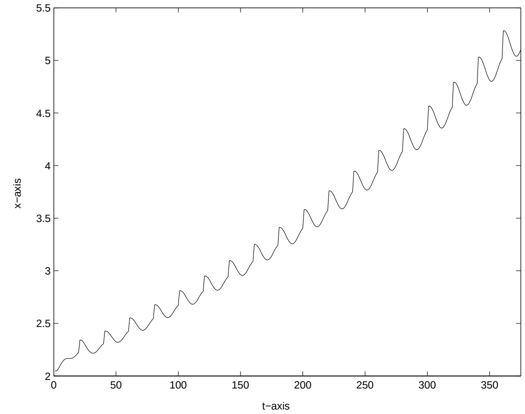

Figure 2

Case 2. Choose $b = 0.05$, $\tau = 1$ and $C = 0$. We check the conditions for *Theorem 4.1* as follows: $2p(t) \geq (1+\mu)L|q(t)|$ is satisfied as in *Case 1* for $\mu = 4/3$. At the same time, since $(1+b)^2 = (1+0.05)^2 = 1.010025 > 3/4 = 1/\mu$, from (4.9) we have

$$\begin{aligned} Q &= (1+b)^2 S \\ &= (1+b)^2 \left\{ \left[(1+b)^2 - 1/\mu\right] \cdot \exp\left(-\int_0^{\tau/2} M(s)ds\right) + 1/\mu \right\} \\ &= 1.010025 \times \left\{ (1.010025 - 3/4) \cdot \exp\left(-\int_0^{1/2} 2\sin(2\pi s)ds\right) + 3/4 \right\} \\ &= 1.010025 \times \left(0.260025 \times e^{-2/\pi} + 3/4\right) \approx 0.8965 < 1. \end{aligned}$$

Hence all the conditions for *Theorem 4.1* are satisfied. So the solution $x(t)$ of problem (5.1) has the following asymptotic weighted periodicity:

$$\lim_{t\to\infty} |x(t) - 1.05x(t-1)| = 0.$$

Since the amplitude of $x(t)$ may get nearly to be 1.05 times after a unit time t, so $x(t)$ gets larger and larger as t increases just as shown in *Figure 2*.

Case 3. Choose $b = -2$, $\tau = 1$ and $C = 0$. We check the conditions for *Theorem 4.2* as follows: Since $(1+b)^2 = (1-2)^2 = 1 > 3/4 = 1/\mu$, from the denotations in (4.9),

$$\begin{aligned} Q &= (1+b)^2 S \\ &= (1+b)^2 \left\{ \left[(1+b)^2 - 1/\mu\right] \cdot \exp\left(-\int_0^{\tau/2} M(s)ds\right) + 1/\mu \right\} \\ &= 1 \times \left\{ (1 - 3/4) \cdot \exp\left(-\int_0^{1/2} 2\sin(2\pi s)ds\right) + 3/4 \right\} = \frac{1}{4}e^{-2/\pi} + \frac{3}{4} < 1. \end{aligned}$$

Hence all the conditions for *Theorem 4.2* are satisfied. So the solution $x(t)$ of problem (5.1) has the following asymptotic weighted periodicity:

$$\lim_{t\to\infty} |x(t) - x(t-2)| = 0.$$

Just as indicated by *Remark 4.2* for the case $f(0) = 0$ that (5.4) also includes the case $\lim_{t\to\infty} |x(t)| = 0$. Accordingly, the simulation is given by *Figure 3*.

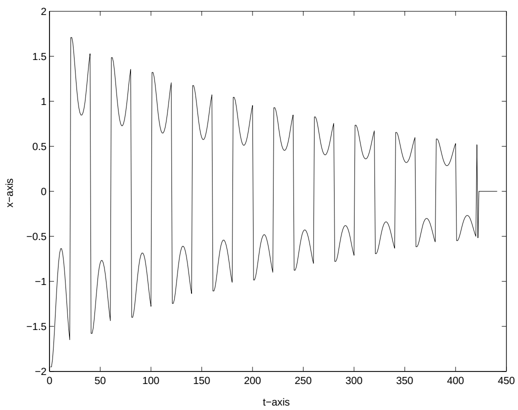

Figure 3

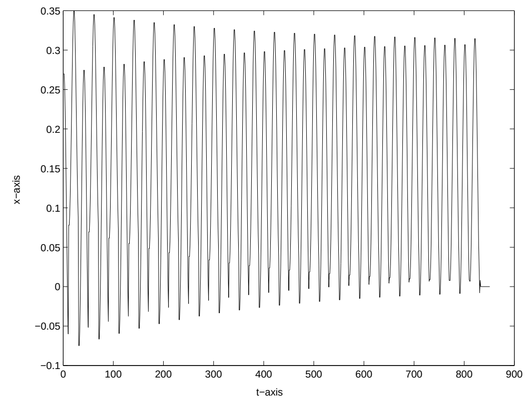

Figure 4

Case 4. Choose $b = -2$, $\tau = 1$ and $C = 1$. At this time we also have $L = 1$ and (5.4) also holds true. Since at this time $f(0) = 1 \neq 0$, we have an asymptotic periodic solution $x(t)$ in the normal sense (see *Figure 4*).

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