EXISTENCE, UNIQUENESS, AND CONTROLLABILITY RESULTS FOR NEUTRAL FSDES IN HILBERT SPACES

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ABSTRACT. We establish results concerning the global existence, uniqueness, and controllability of mild solutions for a neutral functional stochastic differential equations with variable delay in a real separable Hilbert space. The results are obtained by imposing a so-called Carathéodory condition on the nonlinearities, which is weaker than the classical Lipschitz condition. Examples illustrating the applicability of the general theory are also provided.

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1. INTRODUCTION

In this paper we are concerned with the existence, uniqueness, and controllability of mild solutions to neutral functional stochastic differential equations with variable time delay of the form

$$
\begin{aligned}
&d\left[X\left(t\right)+g\left(t,X\left(\rho\left(t\right)\right)\right)\right]=\left[AX\left(t\right)+f\left(t,X\left(\rho\left(t\right)\right)\right)\right]dt \\
&\qquad +\sigma\left(t,X\left(\rho\left(t\right)\right)\right)dW\left(t\right),\ t\in[0,T] \\
&X\left(t\right)=\phi\left(t\right),\quad t\in\left[-r,0\right],
\end{aligned}
\tag{1.1}
$$

where A is the infinitesimal generator of a strongly continuous semigroup of bounded linear operators $\{S\left(t\right):t\geq 0\}$ in a separable Hilbert space H, $W\left(t\right)$ is a Wiener process on a separable Hilbert space K, $X(t)$ is a state process, $g:[0,T]\times H\rightarrow H$, $f:[0,T]\times H\rightarrow H$ and $\sigma:[0,T]\times H\rightarrow L_2^0$ are given functions to be specified later, $\rho:[0,\infty)\rightarrow[-r,\infty)$ is a suitable delay function, $\phi:[-r,0]\times\Omega\rightarrow H$ is the initial datum.

For ordinary SDEs there are some articles which have dealt with existence and uniqueness of solution under non-Lipschitz coefficients, see Yamada [6], Taniguchi [25], Rodkina [24]. Recently, many results on the existence and uniqueness of mild solutions of various type of evolution equations in Hilbert spaces have been obtained without assuming a Lipschitz condition, see [3], [7], [8], [9], [27], [22]. Qualitative theory of neutral stochastic differential (delay) equations in finite dimensional spaces have recently been studied intensively, see Kolmanovskii V.B. and Nosov V.R. [10], and Mao X. [17]- [21], Kolmanovskii V. *et al.* [11], Liu K., and Xia X. [12]. Recently Mahmudov [15] studied existence of mild solutions to neutral stochastic differential equations in Hilbert spaces. On the other hand controllability concepts of stochastic equations are studied in the papers [2], [5], [13], [14], [16].

So far little is known about the neutral stochastic functional evolution equations in Hilbert spaces and the aim of this paper is to close this gap. In this paper motivated by the above mentioned papers we will study the existence, uniqueness, and controllability of solutions of equation (1.1) with non-Lipschitz coefficients by using a Picard type iteration. In particular we can see that the Lipschitz condition is a special case of the proposed conditions and to all appearance the result is new even when f and σ satisfy the Lipschitz condition.

The results presented in the current manuscript constitute a continuation and generalization of existence, uniqueness, and controllability results from [3], [7], [8], [9], [27], [22], [5], [13], [14], [16] in two ways. Firstly, we incorporate a so-called variable delay function (1.1). Secondly, more importantly, we replace the Lipschitz growth conditions by more general Caratheódory-type conditions of the type introduced by [24] and subsequently adapted in [3], [9], [7], [15].

The following is the outline of the paper. First, we make precise the necessary notation, function spaces, and definitions, and gather certain preliminary results in Section 2. We then formulate the main results in Section 3, while we devote Section 4 and 5 to the proof of the main results. Section 6 is devoted to a discussion of some concrete example.

2. PRELIMINARIES

In this section we mention few results and notations needed to establish our results. For details, we refer the reader to [4], [23] and the references therein. Throughout this paper, H and K shall denote real separable Hilbert spaces with respective norms $\|\cdot\|$ and $\|\cdot\|_K$. Let $(\Omega, \mathfrak{F}_T, P)$ be a complete probability space equipped with a normal filtration $\{\mathfrak{F}_t : t \geq 0\}$ generated by the Q-Wiener process W on $(\Omega, \mathfrak{F}_T, \mathbf{P})$ with the linear bounded covariance operator such that $\mathrm{tr}Q < \infty$. We assume that there exists a complete orthonormal system $\{e_k\}$ in K, a bounded sequence of non-negative real numbers λ_k such that $Qe_k = \lambda_k e_k$, $k = 1, 2, ...$ and a sequence $\{\beta_k\}$ of independent Brownian motions such that

$$\langle W(t), e\rangle = \sum_{k=1}^{\infty} \sqrt{\lambda_k} \langle e_k, e\rangle \beta_k(t), \ e \in K, \ t \in [0, T].$$

Let $L_2^0 = L_2\left(Q^{1/2}K, H\right)$ be the space of all Hilbert-Schmidt operators from $Q^{1/2}K$ to H with the inner product $\langle \Psi, \Phi \rangle_{L_2^0} = tr\left[\Psi Q \Phi^*\right]$. $L^p\left(\Omega, \mathfrak{F}_T, H\right)$ is the Hilbert space of all $\mathfrak{F}_T$-measurable square integrable variables with values in a Hilbert space H.

$L^p_{\mathfrak{F}}([0,T], H)$ is the Hilbert space of all p-integrable and $\mathfrak{F}_t$ -adapted processes with values in H. We recall that f is said to be $\mathfrak{F}_t$ -adapted if $f(t,\cdot) : \Omega \to H$ is $\mathfrak{F}_t$ -measurable, a.e. $t \in [0,T]$. Let $\mathfrak{H}_p$ denote the Banach space of all H-valued $\mathfrak{F}_t$ -adapted processes $X(t,\omega) : [0,T] \times \Omega \to H$ which are continuous in t for a.e. fixed $\omega \in \Omega$ and satisfy

$$\|X\|_{\mathfrak{H}_p} = \left\{\mathbf{E}\left(\sup_{t\in[0,T]} \|X(t,\omega)\|^p\right)\right\}^{1/p} < \infty, \quad p > 2.$$

p and r are conjugate indices: $\dfrac{1}{p} + \dfrac{1}{r} = 1$. For brevity, we suppress the dependence of all mappings on ω throughout the manuscript.

Let A be the infinitesimal generator of an analytic semigroup $S(t)$ in H. If A is the infinitesimal generator of an analytic semigroup then $A - \alpha I$ is invertible and generates a bounded analytic semigroup. for $\alpha > 0$ large enough. This allows to reduce the general case in which A is the infinitesimal generator of an analytic semigroup to the case in which semigroup is bounded and the generator is invertible. Hence for convenience, we suppose that $\|S(t)\| \leq M$ for $t \geq 0$ and $0 \in \rho(A)$, where $\rho(A)$ is the resolvent set of A. It follows that for $0 < \alpha \leq 1$, $(-A)^\alpha$ can be defined as a closed linear invertible operator with its domain $D(-A)^\alpha$ being dense in H. We denote by H_α the Banach space $D(-A)^\alpha$ endowed with norm $\|x\|_\alpha = \|(-A)^\alpha x\|$ which is equivalent to the graph norm of $(-A)^\alpha$.

Lemma 2.1. [23] *The following properties hold.*

1. *If $0 < \beta < \alpha \leq 1$ then $H_\alpha \subset H_\beta$ and the imbedding is compact whenever the resolvent operator of A is compact.*
2. *For every $0 < \alpha \leq 1$ there exists $C_\alpha > 0$ such that*

$$\|(-A)^\alpha S(t)\| \leq \frac{C_\alpha}{t^\alpha}, t > 0.$$

Lemma 2.2. [4] *Let $p > 2$, $T > 0$ and let Φ be an L_2^0-valued, predictable process such that $\mathbf{E}\int_0^T \|\Phi(s)\|^p_{L_2^0} ds < \infty$. Then there exists a constant $M_T > 0$ such that*

$$\mathbf{E}\sup_{t\in[0,T]} \left\|\int_0^t S(t-s)\Phi(s)\,\mathrm{d}W(s)\right\|^p \leq M_T \mathbf{E}\int_0^T \|\Phi(s)\|^p_{L_2^0}\,\mathrm{d}s.$$

Lemma 2.3. *Let u, ψ and χ be real continuous functions defined on $[a,b]$, $\chi(t) \geq 0$ for $t \in [a,b]$. We suppose that on $[a,b]$ we have the inequality*

$$u(t) \leq \psi(t) + \int_a^t \chi(s)\,u(s)\,\mathrm{d}s.$$

If ψ is differentiable, then

$$u(t) \leq \psi(a) \exp\left(\int_a^t \chi(s)\,\mathrm{d}s\right) + \int_a^t \exp\left(\int_s^t \chi(r)\,dr\right) \psi'(s)\,\mathrm{d}s$$

for all $t \in [a,b]$.

3. MAIN RESULTS

The following are the main assumptions assumed in the manuscript.

(A1): A is the infinitesimal generator of an analytic semigroup $\{S(t), t > 0\}$ on H.

(A2): $(f, \sigma) : [0,T] \times H \to H \times L_2^0$ are $\mathfrak{F}_t$-measurable mappings satisfying:

(i): There exist some $K : [0,\infty) \to [0,\infty)$ such that

(a): K is continuous, nondecreasing, and concave,

(b): $\|f(t,x)\|^p + \|\sigma(t,x)\|_{L_2^0}^p \leq K(\|x\|^p)$, for all $(t,x) \in [0,T] \times H$.

(ii): There exist some $N : [0,\infty) \to [0,\infty)$ such that

(a): N is continuous, nondecreasing and concave, and $N(0) = 0$,

(b): $\|f(t,x) - f(t,y)\|^p + \|\sigma(t,x) - \sigma(t,y)\|_{L_2^0}^p \leq N(\|x-y\|^p)$, for all $(t,x), (t,y) \in [0,T] \times H$.

(A3): The function N of **(A2)**(ii) is such that if a nonnegative, continuous function $z(t)$ satisfies $z(0) = 0$ and

$$z(t) \leq D \int_0^t N(z(s))\,ds,$$

for all $t \in [0,T]$, where $D > 0$, then $z(t) = 0$, for all $t \in [0,T]$.

(A4): For any fixed $T > 0$, $\beta > 0$ the initial-value problem

$$u'(t) = \beta K(u),\ u(0) = u_0 \geq 0, \tag{3.1}$$

has a global solution on $[0,T]$.

(A5): $\rho : [0,\infty) \to [-r,\infty)$ is a continuously differentiable function of delay satisfying the conditions that

$$\rho'(t) \geq 1, \ -r \leq \rho(t) \leq t, \ \text{ for } r > 0 \text{ and } t \geq 0.$$

(Observe that there exists a constant $k > 0$ such that $\rho^{-1}(t) \leq t + k$, for all $t \geq -r$.)

(A6): The function $\phi(t) : [-r,0] \times \Omega \to H$ is an $\mathfrak{F}_0$-measurable random variable independent of W with almost surely continuous paths.

(A7): There exist positive constants $\frac{1}{p} < \beta < 1$, l, M_g such that g is H_β-valued, $(-A)^\beta g$ is continuous and

$$\left\|(-A)^\beta g(t,x)\right\|^p \leq l\left(\|x\|^p + 1\right), \qquad (t,x) \in [0,T] \times H,$$

$$\left\|(-A)^{\beta} g\left(t, x_{1}\right)-(-A)^{\beta} g\left(t, x_{2}\right)\right\|^{p} \leq M_{g}\left\|x_{1}-x_{2}\right\|^{p}, \quad\left(t, x_{i}\right) \in[0, T] \times H.$$

(A8): Constants M_g and l satisfy the following inequalities

$$4^{p-1}\left\|(-A)^{-\beta}\right\|^{p} M_{g}<1, \quad 5^{p-1}\left\|(-A)^{-\beta}\right\|^{p} l<1.$$

(A9): For each $0 \leq t < T$, the operator $\alpha\left(\alpha I+\Gamma_{t}^{T}\right)^{-1} \rightarrow 0$ as $\alpha \rightarrow 0^{+}$ in the strong operator topology, where $\Gamma_{t}^{T}=\int_{t}^{T} S(T-s) B B^{*} S^{*}(T-s) d s$ is the controllability Grammian. Observe that the linear deterministic system

$$x^{\prime}(t)=A x(t)+B u(t), \quad 0 \leq t \leq T,$$
$$x(0)=x_{0} \in H,$$

corresponding to (1.1) is approximately controllable on $[t, T]$ if and only if the operator $\alpha\left(\alpha I+\Gamma_{t}^{T}\right)^{-1} \rightarrow 0$ strongly as $\alpha \rightarrow 0^{+}$ (see [13], [14]).

Definition 3.1. A continuous stochastic process $X:[-r, T] \times \Omega \rightarrow H$ is a mild solution of (1.1) if the following conditions are satisfied:

(i) $X(t)$ is measurable and $\mathfrak{F}_t$-adapted, for all $-r \leq t \leq T$,
(ii) $\int_0^T \|X(s)\|^2 ds < \infty, \quad a.s.,$
(iii) X satisfies the integral equation

$$\begin{aligned} X(t)= & S(t)(X(0)+g(0, \phi))-g(t, X(\rho(t))) \\ & -\int_{0}^{t} A S(t-s) g(s, X(\rho(s))) d s+\int_{0}^{t} S(t-s) f(s, X(\rho(s))) d s \\ & +\int_{0}^{t} S(t-s) \sigma(s, X(\rho(s))) d W(s), 0 \leq t \leq T, \\ X(t)= & \phi(t), \quad t \in[-r, 0] . \end{aligned} \tag{3.2}$$

Theorem 3.2 (Existence and uniqueness). *If the coefficients f, g, σ satisfy (A1)-(A8), then the equation (1.1) has a unique mild solution on $[0,T]$.*

Corollary 3.3. *Suppose that (A7) and (A8) are satisfied. Further suppose that the following conditions are satisfied:*

(C1) $\|f(t, x)-f(t, y)\|^{p}+\|\sigma(t, x)-\sigma(t, y)\|_{L_{2}^{0}}^{p} \leq \alpha(t) \rho\left(\|x-y\|^{p}\right), \; p>2,$
(C2) $\|f(t, 0)\|,\|\sigma(t, 0)\|_{L_{2}^{0}} \in L^{p}\left([0, T], R^{+}\right)$ *for all* $t \in [0,T]$ *and* $x, y \in H$, *where* $\alpha(t) \geq 0$ *is locally integrable and* $\rho(u)$ *is a continuous concave nondecreasing function from* R_+ *to* R_+ *such that* $\rho(0)=0$, $\rho(u)>0$ *for* $u>0$ *and* $\int_{0+} \frac{1}{\rho(u)}=\infty$. *Then the equation (1.1) has a unique mild solution on* $[0,T]$.

Consider the following control system

$$X(t)=S(t)(X(0)+g(0, X(0)))-g(t, X(t))$$

$$-\int_0^t AS(t-s)\,g(s,X(s))\,ds+\int_0^t S(t-s)\left[Bu(s)+f(s,X(\rho(s)))\right]ds$$
$$+\int_0^t S(t-s)\,\sigma(s,X(\rho(s)))\,dW(s)\,,0\le t\le T, \tag{3.3}$$
$$X(t)=\phi(t)\,,\quad t\in[-r,0]\,,$$

where $u(t)$ is a control, B is a bounded linear operator from a separable Hilbert space U to H,

Definition 3.4. The system (3.3) is approximately controllable on $[0,T]$ if $\overline{R(T)}=L^p(\Omega,\mathfrak{F}_T,H)$, where

$$R(T)=\{X(T;u):X(t,u)\text{ is a mild solution of (1.1)}$$
$$\text{corresponding to }u\in L^p_{\mathfrak{F}}(0,T;U)\Big\}\,.$$

Theorem 3.5 (Controllability)**.** *Assume that conditions (A1)-(A9) hold. If the functions f, σ, and $(-A)^{\beta}g$ are uniformly bounded on their appropriate domains, the function $g(T,x)$ and the semigroup $\{S(t):t>0\}$ is compact, then the system (1.1) is approximately controllable on $[0,T]$.*

4. EXISTENCE AND UNIQUENESS RESULTS

4.1. **Simple Equation.** For any $(f,\sigma)\in L^p_{\mathfrak{F}}([0,T],H)\times L^p_{\mathfrak{F}}([0,T],L_2^0)$ look at the following equation

$$\mathrm{d}\left[X(t)+g(t,X(\rho(t)))\right]=\left[AX(t)+f(t)\right]\mathrm{d}t+\sigma(t)\,\mathrm{d}W(t)\,,$$
$$X(t)=\varphi(t)\,,\quad t\in[-r,0] \tag{4.1}$$

and define the following operator $\Psi:\mathfrak{H}_p\to\mathfrak{H}_p$

$$(\Psi X)(t)=S(t)\left(X(0)+g(0,X(0))\right)-g(t,X(\rho(t)))$$
$$-\int_0^t AS(t-s)\,g(s,X(\rho(s)))\,ds+\int_0^t S(t-s)\,f(s)\,ds$$
$$+\int_0^t S(t-s)\,\sigma(s)\,dW(s)$$
$$=S(t)\left(X(0)+g(0,X(0))\right)+\sum_{i=1}^{4}I_i(t)\,. \tag{4.2}$$

Lemma 4.1. *Under the condition (A7) the equation (4.1) has a unique mild solution in $\mathfrak{H}_p$ if*

$$4^{p-1}\left\|(-A)^{-\beta}\right\|^p M_g<1.$$

Proof. To see that $\Psi(\mathfrak{H}_p) \subset \mathfrak{H}_p$, let $X \in \mathfrak{H}_p$. Standard computations involving the Hōlder inequality yield the following estimates:

$$\mathbf{E} \sup_{s\in[0,t]} \|S(s)(X(0)+g(0,X(0)))\|^p \leq M_S^p \mathbf{E}\|X(0)+g(0,X(0))\|^p,$$

$$\mathbf{E}\|I_1\|_t^p \leq \left\|(-A)^{-\beta}\right\|^p l\left[\mathbf{E}\|X(\rho)\|_t^p+1\right],$$

$$\begin{aligned}\mathbf{E}\|I_2\|_t^p &\leq \mathbf{E}\left(\int_0^t \left\|(-A)^{1-\beta} S(t-s)(-A)^{\beta} g(s,X(\rho(s)))\right\| \mathrm{d}s\right)^p \\ &\leq \left(\int_0^t \left\|(-A)^{1-\beta} S(t-s)\right\|^r \mathrm{d}s\right)^{p/r} \int_0^t \mathbf{E}\left\|(-A)^{\beta} g(s,X(\rho(s)))\right\|^p \mathrm{d}s \\ &\leq \left(\int_0^t \frac{C_{1-\beta}^r}{(t-s)^{(1-\beta)r}}\mathrm{d}s\right)^{p/r} l\int_0^t \left[\mathbf{E}\|X(\rho(s))\|^p+1\right]\mathrm{d}s \\ &\leq \left(\frac{C_{1-\beta}^r T^{r(\beta-1)+1}}{r(\beta-1)+1}\right)^{p/r} l\int_0^t \left[\mathbf{E}\|X(\rho(s))\|^p+1\right]\mathrm{d}s, \quad \beta > 1-\frac{1}{r}=\frac{1}{p},\end{aligned}$$

$$\mathbf{E}\|I_3\|_t^p \leq T^{p-1} M_S^p \int_0^t \mathbf{E}\|f(s)\|^p \mathrm{d}s,$$
$$\mathbf{E}\|I_4\|_t^p \leq M_T \int_0^t \mathbf{E}\|\sigma(s)\|_{L_2^0}^2 \mathrm{d}s.$$

Thus the above inequalities together imply that $\mathbf{E}\|\Psi X\|_T^p < \infty$. Since $\mathfrak{F}_t$-measurability of $(\Psi X)(t)$ is easily verified, we conclude that Ψ is well defined. Next, we prove that Ψ has a unique fixed point. Indeed, for any $X, Y \in \mathfrak{H}_p$, standard computations yield

$$\begin{aligned}\mathbf{E}\|\Psi X-\Psi Y\|_t^p &\leq 4^{p-1}\left\|(-A)^{-\beta}\right\|^p M_g \mathbf{E}\|X-Y\|_t^p \\ &+4^{p-1}\left(\frac{C_{1-\beta}^r t^{r(\beta-1)+1}}{r(\beta-1)+1}\right)^{p/r} M_g \int_0^t \mathbf{E}\|X(\rho(s))-Y(\rho(s))\|^p \mathrm{d}s \\ &\leq \left(4^{p-1}\left\|(-A)^{-\beta}\right\|^p M_g + 4^{p-1}\left(\frac{C_{1-\beta}^r t^{r(\beta-1)+1}}{r(\beta-1)+1}\right)^{p/r} M_g^p t\right)\mathbf{E}\|X-Y\|_t^p \\ &= \gamma(t)\,\mathbf{E}\|X-Y\|_t^p.\end{aligned}$$

By assumption **(A8)** the first term of $\gamma(t)$ is less that 1. Then there exists $0 < T_1 \leq T$ such that $0 < \gamma(T_1) < 1$ and the operator Ψ is a contraction on $\mathfrak{H}_p$ and therefore has a unique fixed point, which is a mild solution of (4.1) of on $[0, T_1]$. This procedure can be repeated in order to extend the solution to the entire interval $[0, T]$ in finitely many steps, thereby completing the proof. □

Next we define the operator $\Phi : L_{\mathfrak{F}}^p([0,T], H) \times L_{\mathfrak{F}}^p([0,T], L_2^0) \rightarrow \mathfrak{H}_p$ as follows

$$\Phi(f,\sigma) = X,$$

$$X(t) = S(t)(X(0) + g(0, X(0))) - g(t, X(\rho(t))) - \int_0^t AS(t-s)\, g(s, X(\rho(s)))\, ds + \int_0^t S(t-s) f(s)\, ds + \int_0^t S(t-s)\, \sigma(s)\, dW(s), \tag{4.3}$$

where X is a mild solution of the equation (4.1).

Lemma 4.2. *Under the conditions (A1)-(A8) the operator Φ satisfies the following: there exist positive constants $M_\Phi, \overline{M}_\Phi, D_0$ such that*

$$\mathbf{E}\|\Phi(f_1,\sigma_1) - \Phi(f_2,\sigma_2)\|_t^p \leq M_\Phi \int_0^t \left[\mathbf{E}\|f_1(s) - f_2(s)\|^p + \mathbf{E}\|\sigma_1(s) - \sigma_2(s)\|_{L_2^0}^p\right] \mathrm{d}s, \tag{4.4}$$

$$\mathbf{E}\|\Phi(f,\sigma)\|_t^p \leq D_0 + \overline{M}_\Phi \int_0^t \left[\mathbf{E}\|f(s)\|^p + \mathbf{E}\|\sigma(s)\|_{L_2^0}^p\right] \mathrm{d}s, \tag{4.5}$$

for all $(f,\sigma), (f_1,\sigma_1), (f_2,\sigma_2) \in L^p_{\mathfrak{F}}([0,T], H) \times L^p_{\mathfrak{F}}([0,T], L_2^0)$, $p > 2$.

Proof. Indeed, for any $(f_1,\sigma_1), (f_2,\sigma_2) \in L^p_{\mathfrak{F}}([0,T], H) \times L^p_{\mathfrak{F}}([0,T], L_2^0)$, such that

$$\Phi(f_1,\sigma_1) = X = S(t)(X(0) + g(0, X(0))) + \sum_{i=1}^4 I_i^1,$$

$$\Phi(f_2,\sigma_2) = Y = S(t)(X(0) + g(0, X(0))) + \sum_{i=1}^4 I_i^2$$

we have

$$\mathbf{E}\|\Phi(f_1,\sigma_1) - \Phi(f_2,\sigma_2)\|_t^p \leq 4^{p-1}\sum_{i=1}^4 \mathbf{E}\left\|I_i^1 - I_i^2\right\|_t^p. \tag{4.6}$$

Standard computations yield

$$\mathbf{E}\left\|I_1^1 - I_1^2\right\|_t^p \leq \left\|(-A)^{-\beta}\right\|^p M_g \mathbf{E}\|X - Y\|_t^p, \tag{4.7}$$

$$\begin{aligned}
&\mathbf{E}\left\|I_2^1 - I_2^2\right\|_t^p \\
&\leq \mathbf{E}\left(\int_0^t \left\|(-A)^{1-\beta} S(t-s)(-A)^\beta [g(s, X(\rho(s))) - g(s, Y(\rho(s)))]\right\| \mathrm{d}s\right)^p \\
&\leq \left(\int_0^t \left\|(-A)^{1-\beta} S(t-s)\right\|^r \mathrm{d}s\right)^{p/r} \\
&\times \int_0^t \mathbf{E}\left\|(-A)^\beta [g(s, X(\rho(s))) - g(s, Y(\rho(s)))]\right\|^p \mathrm{d}s
\end{aligned}$$

$$\leq \left(\int_0^t \frac{C_{1-\beta}^r}{(t-s)^{(1-\beta)r}} \mathrm{d}s\right)^{p/r} M_g \int_0^t \mathbf{E} \left\| X\left(\rho\left(s\right)\right) - Y\left(\rho\left(s\right)\right)\right\|^p \mathrm{d}s$$

(4.8)

$$\leq \left(\frac{C_{1-\beta}^r T^{r(\beta-1)+1}}{r\left(\beta-1\right)+1}\right)^{p-1} M_g \int_0^t \mathbf{E} \left\| X\left(\rho\left(s\right)\right) - Y\left(\rho\left(s\right)\right)\right\|^p \mathrm{d}s, \quad \beta > 1 - \frac{1}{r} = \frac{1}{p},$$

$$\mathbf{E} \left\| I_3^1 - I_3^2 \right\|_t^p \leq T^{p-1} M_S^p \int_0^t \mathbf{E} \left\| f_1\left(s\right) - f_2\left(s\right)\right\|^p \mathrm{d}s, \tag{4.9}$$

$$\mathbf{E} \left\| I_4^1 - I_4^2 \right\|_t^p \leq M_T \int_0^t \mathbf{E} \left\| \sigma_1\left(s\right) - \sigma_2\left(s\right)\right\|_{L_2^0}^p \mathrm{d}s. \tag{4.10}$$

Using (4.7)-(4.10), along with (4.6), gives rise to

$$\begin{aligned}
&\mathbf{E} \left\| X - Y \right\|_t^p \\
&\leq 4^{p-1} \left\| (-A)^{-\beta} \right\|^p M_g \mathbf{E} \left\| X - Y \right\|_t^p \\
&+ 4^{p-1} \left(\frac{C_{1-\beta}^r T^{r(\beta-1)+1}}{r\left(\beta-1\right)+1}\right)^{p/r} M_g \int_0^t \mathbf{E} \left\| X\left(\rho\left(s\right)\right) - Y\left(\rho\left(s\right)\right)\right\|^p \mathrm{d}s \\
&+ 4^{p-1} T^{p-1} M_S^p \int_0^t \mathbf{E} \left\| f_1\left(s\right) - f_2\left(s\right)\right\|^p \mathrm{d}s + 4^{p-1} M_T \int_0^t \mathbf{E} \left\| \sigma_1\left(s\right) - \sigma_2\left(s\right)\right\|_{L_2^0}^p \mathrm{d}s
\end{aligned}$$

or, equivalently

$$\mathbf{E} \left\| X - Y \right\|_t^p \leq D_1 \int_0^t \mathbf{E} \left\| X - Y \right\|_r^p \mathrm{d}r$$

$$+ D_2 \int_0^t \mathbf{E} \left\| f_1\left(s\right) - f_2\left(s\right)\right\|^p \mathrm{d}s + D_3 \int_0^t \mathbf{E} \left\| \sigma_1\left(s\right) - \sigma_2\left(s\right)\right\|_{L_2^0}^p \mathrm{d}s,$$

where

$$\begin{aligned}
&C_1 = 4^{p-1} \left\| (-A)^{-\beta} \right\|^p M_g, \quad C_2 = 4^{p-1} \left(\frac{C_{1-\beta}^r T^{r(\beta-1)+1}}{r\left(\beta-1\right)+1}\right)^{p/r} \\
&D_1 = C_2 M_g / \left(1 - C_1\right), \quad D_2 = 4^{p-1} T^{p-1} M_S^p / \left(1 - C_1\right), \\
&D_3 = 4^{p-1} M_T / \left(1 - C_1\right).
\end{aligned}$$

By Lemma 2.3

$$\begin{aligned}
&\mathbf{E} \left\| X - Y \right\|_t^p \\
&\leq \int_0^t \exp\left(D_1\left(t-s\right)\right) \left[D_2 \mathbf{E} \left\| f_1\left(s\right) - f_2\left(s\right)\right\|^p + D_3 \mathbf{E} \left\| \sigma_1\left(s\right) - \sigma_2\left(s\right)\right\|_{L_2^0}^p\right] \mathrm{d}s.
\end{aligned}$$

Now (4.4) follows with $M_\Phi = \exp\left(D_1 T\right)\left(D_2 + D_3\right)$. Similarly, standard computations yield $\left(\Phi\left(f, \sigma\right) = X\right)$

$$\mathbf{E} \left\| \Phi\left(f, \sigma\right)\right\|_t^p \leq \overline{C}_1 + \overline{C}_2 \mathbf{E} \left\| X \right\|_t^p + \overline{C}_3 \int_0^t \mathbf{E} \left\| X \right\|_s^p \mathrm{d}s$$

$$+\overline{C}_4\int_0^t \mathbf{E}\left\|f(s)\right\|^p \mathrm{d}s+\overline{C}_5\int_0^t \mathbf{E}\left\|\sigma(s)\right\|^p_{L_2^0}\mathrm{d}s, \tag{4.11}$$

where

$$\overline{C}_1 = 5^{p-1}M_S^p\mathbf{E}\left\|x_0+g(0,x_0)\right\|^p + 5^{p-1}\left\|(-A)^{-\beta}\right\|^p l$$
$$+5^{p-1}\left(\frac{C_{1-\beta}^r T^{r(\beta-1)+1}}{r(\beta-1)+1}\right)^{p/r} Tl,$$
$$\overline{C}_2 = 5^{p-1}\left\|(-A)^{-\beta}\right\|^p l,\quad \overline{C}_3 = 5^{p-1}\left(\frac{C_{1-\beta}^r T^{r(\beta-1)+1}}{r(\beta-1)+1}\right)^{p/r} l,$$
$$\overline{C}_4 = 5^{p-1}M_S^pT^{p-1},\quad \overline{C}_5 = 5^{p-1}M_T.$$

Now we solve the inequality (4.11) for $\mathbf{E}\left\|X\right\|_t^p$ and apply to the obtained inequality the Gronwall lemma to get (4.5) with

$$D_1=\frac{\overline{C}_1}{1-\overline{C}_2},\quad D_2=\frac{\overline{C}_3}{1-\overline{C}_2},\quad D_3=\frac{\overline{C}_4}{1-\overline{C}_2},\quad D_4=\frac{\overline{C}_5}{1-\overline{C}_2},$$
$$\overline{M}_\Phi=\exp(D_2T)\max(D_3,D_4),\quad D_0=D_1\exp(D_2T).$$

□

4.2. **Proof of Theorem 3.2.** To prove the result concerning the existence and uniqueness of mild solutions to (1.1) we now construct an approximation sequence using a Picard type iteration. For any fixed $T>0$, let X_0 be a solution of (4.3) with $f=0$, $\sigma=0$ defined by

$$X_0(t)=S(t)(X(0)+g(0,\phi))-g(t,X_0(\rho(t)))$$
$$-\int_0^t AS(t-s)\,g(s,X_0(\rho(s)))\,ds,$$

and let X_n be a sequence defined recursively by

$$\begin{aligned}X_n(t)&=S(t)(X(0)+g(0,\phi))-g(t,X_n(\rho(t)))\\&-\int_0^t AS(t-s)\,g(s,X_n(\rho(s)))\,\mathrm{d}s+\int_0^t S(t-s)\,f(s,X_{n-1}(\rho(s)))\,\mathrm{d}s\\&+\int_0^t S(t-s)\,\sigma(s,X_{n-1}(\rho(s)))\,\mathrm{d}W(s),\quad t\in[0,T].\end{aligned}\tag{4.12}$$

By Lemma 4.1 the equation (1.1) has a unique solution and it is clear that $X_n = \Phi\left(f(\cdot,X_{n-1}(\rho(\cdot))),\sigma(\cdot,X_{n-1}(\rho(\cdot)))\right)$, where Φ is defined by (4.3).

Lemma 4.3. *Under the conditions (A1)-(A6) the operator $\Pi:\mathfrak{H}_p\to\mathfrak{H}_p$, where*

$$(\Pi X)(t)=\left[\Phi\left(f(\cdot,X(\rho(\cdot))),\sigma(\cdot,X(\rho(\cdot)))\right)\right](t),$$

is well defined and there are positive constants $M_\Phi, \overline{M}_\Phi, D_0$ *such that*

$$\mathbf{E}\left\|\Pi X\right\|_t^p \leq D_0 + \overline{M}_\Phi \int_0^t K\left(\mathbf{E}\left\|X\right\|_s^p\right) \mathrm{d}s, \tag{4.13}$$

$$\mathbf{E}\left\|\Pi X - \Pi Y\right\|_t^p \leq M_\Phi \int_0^t N\left(\mathbf{E}\left\|X - Y\right\|_s^p\right) \mathrm{d}s, \tag{4.14}$$

for all $t \in [0, T]$.

Proof. By Lemma 4.2 we have

$$\begin{aligned}
&\mathbf{E}\sup_{s\in[0,t]} \left\|(\Pi X)(s) - (\Pi Y)(s)\right\|^p \\
&\leq M_\Phi \int_0^t \Big[\mathbf{E}\left\|f(s, X(\rho(s))) - f(s, Y(\rho(s)))\right\|^p \\
&+ \mathbf{E}\left\|\sigma(s, X(\rho(s))) - \sigma(s, Y(\rho(s)))\right\|_{L_2^0}^p\Big] \mathrm{d}s \\
&\leq M_\Phi \int_0^t N\left(\mathbf{E}\left\|X(\rho(s)) - Y(\rho(s))\right\|^p\right) \mathrm{d}s \\
&\leq M_\Phi \int_0^t N\left(\mathbf{E}\left\|X - Y\right\|_s^p\right) \mathrm{d}s.
\end{aligned}$$

So (4.14) is proved. (4.13) can be proved in a similar manner. The proof is completed. □

Lemma 4.4. *Under the conditions (A1) and (A2), the sequence* $\{X_n : n \geq 0\}$ *satisfies the following inequality for all* $0 \leq t \leq T$*:*

$$\mathbf{E}\left\|X_n\right\|_t^p \leq u(t). \tag{4.15}$$

Proof. It follows from Lemma 4.3 that

$$\mathbf{E}\left\|X_n\right\|_t^p \leq M_1 + M_2 \int_0^t K\left(\mathbf{E}\left\|X_{n-1}\right\|_s^p\right) ds, \tag{4.16}$$

where M_1 and $M_2$are positive constants independent of n. Let $u(t)$ be the global solution of the equation (3.1) with $u_0 \geq \max\left(M_1, \mathbf{E}\left\|X_0\right\|_T^p\right)$ and with $u = M_2$. We will establish inequality (4.15) using mathematical induction. To begin, note that for $n = 0$ the inequality (4.15) holds by the definition of u. Indeed, we have

$$\begin{aligned}
u(t) &= u_0 + M_2 \int_0^t K(u(s))\, ds \\
&\geq \max\left(M_1, \mathbf{E}\left\|X_0(t)\right\|_T^p\right) + M_2 \int_0^t K(u(s))\, ds \geq \mathbf{E}\left\|X_0\right\|_T^p.
\end{aligned}$$

Next, suppose that

$$\mathbf{E}\left\|X_{n-1}\right\|_t^p \leq u(t), \quad \text{for all} \quad 0 \leq t \leq T.$$

Then, from (3.1) and (4.16), we conclude that

$$u(t) - \mathbf{E}\left\|X_n\right\|_t^p \geq M_2 \int_0^t \left[K(u(s)) - K\left(\mathbf{E}\left\|X_{n-1}\right\|_s^p\right)\right] ds \geq 0.$$

Hence, (4.15) holds for all n (thanks to **(A2)**). □

Lemma 4.5. *Under the conditions **(A1)** and **(A2)**, $\{X_n : n \geq 1\}$ is a Cauchy sequence in $\mathfrak{H}_p$.*

Proof. Define the sequence of functions $r_n : [0,T] \to \mathbb{R}$ by

$$r_n(t) = \sup_{m \geq n} \mathbf{E}\left\|X_{m+n} - X_n\right\|_t^p, \quad t \in [0,T], \, n \geq 1.$$

Note that for each $n \geq 1$, r_n is well-defined, uniformly bounded, and monotone nondecreasing (in t). Since $\{r_n : n \geq 1\}$is a monotone nonincreasing sequence, for each $t \in [0,T]$, there exists a monotone nondecreasing function $r : [0,T] \to \mathbb{R}$ such that

$$\lim_{n \to \infty} r_n(t) = r(t). \tag{4.17}$$

It follows from Lemma 4.3 that for any $n, m \geq 1$,

$$\mathbf{E}\left\|X_m - X_n\right\|_t^p \leq M_\Phi \int_0^t N\left(\mathbf{E}\left\|X_{m-1} - X_{n-1}\right\|_s^p\right) ds,$$

from which we subsequently obtain

$$r(t) \leq r_n(t) \leq M_\Phi \int_0^t N\left(r_{n-1}(s)\right) ds,$$

for any $n \geq 1$. Using (4.17), together with the Lebesgue dominated convergence theorem, then yields

$$r(t) \leq M_\Phi \int_0^t N(r(s))\, ds.$$

But, $\mathbf{E}\left\|X_{m+n} - X_n\right\|_T^p \leq r_n(T)$ and $\lim_{n\to\infty} r_n(T) = r(T) = 0$. Therefore, $\lim_{m,n\to\infty} \mathbf{E}\left\|X_{m+n} - X_n\right\|_T^p = 0$, so that $\{X_n, n \geq 1\}$ is indeed a Cauchy sequence in $\mathfrak{H}_p$. This completes the proof. □

Theorem 4.6. *If the conditions (A1)-(A7) hold, then (1.1) has a unique mild solution in $\mathfrak{H}_p$.*

Proof. The completeness of $\mathfrak{H}_p$ guarantees the existence of a process X such that

$$\lim_{n\to\infty} \mathbf{E}\left\|X_n - X\right\|_T^p = 0.$$

Further, we may infer from **(A2)** that

$$N\left(\mathbf{E}\left\|X_n - X\right\|_t^p\right) \to N(0) = 0,$$

and hence,

$$\lim_{n\to\infty} \mathbf{E}\left\|\Pi X_n - \Pi X\right\|_t^p = 0.$$

Thus, X is a fixed point of Π which is, in fact, a mild solution to (1.1) on $[0,T]$. Further, if $X,Y\in\mathfrak{H}_p$ are two fixed points of Π, then

$$\mathbf{E}\left\|\Pi X-\Pi Y\right\|_t^p\leq M_\Phi\int_0^t N\left(\mathbf{E}\left\|X-Y\right\|_s^p\right)ds,$$

so that **(A3)** would imply that $\mathbf{E}\left\|\Pi X-\Pi Y\right\|_T^p=0$. Consequently, $X=Y$ in $\mathfrak{H}_p$. Hence, Π has a unique fixed point. □

5. CONTROLLABILITY RESULT (PROOF OF THEOREM 3.5)

It is known that (see [5]) for any $h\in L^p\left(\Omega,\mathfrak{F}_T,H\right)$ there exists $\varphi\in L^p_{\mathfrak{F}}\left(\Omega;L_2\left(0,T;L_2^0\right)\right)$ such that

$$h=\mathbf{E}h+\int_0^T\varphi\left(s\right)dW\left(s\right).$$

Now, using this presentation for any $(\alpha,h,Z)\in(0,\infty)\times L^p\left(\Omega,\mathfrak{F}_T,H\right)\times\mathfrak{H}_p$, we define the control function by

$$\begin{aligned}u^{\alpha}\left(t,Z\right)&=B^*S^*\left(T-t\right)\left(\alpha I+\Gamma_0^T\right)^{-1}\\&\times\left(\mathbf{E}h-S(T)\left(\phi\left(0\right)+g\left(0,\phi\left(0\right)\right)\right)+g\left(T,Z\left(T\right)\right)\right)\\&+B^*S^*\left(T-t\right)\int_0^t\left(\alpha I+\Gamma_s^T\right)^{-1}\varphi\left(s\right)\mathrm{d}W(s)\\&+B^*S^*\left(T-t\right)\int_0^t\left(\alpha I+\Gamma_s^T\right)^{-1}AS(T-s)g\left(s,Z\left(s\right)\right)\mathrm{d}s\\&-B^*S^*\left(T-t\right)\int_0^t\left(\alpha I+\Gamma_s^T\right)^{-1}S(T-s)f\left(s,Z\left(\rho\left(s\right)\right)\right)\mathrm{d}s\\&-B^*S^*\left(T-t\right)\int_0^t\left(\alpha I+\Gamma_s^T\right)^{-1}S(T-s)\sigma\left(s,Z\left(\rho\left(s\right)\right)\right)\mathrm{d}W(s).\end{aligned}\tag{5.1}$$

To present the result concerning the approximate controllability of mild solutions of (3.3), we fix $\alpha>0$ and consider the following equation

$$\begin{aligned}Z(t)&=S\left(t\right)\left(\phi\left(0\right)+g\left(0,\phi\left(0\right)\right)\right)-g\left(t,Z\left(t\right)\right)\\&+\Gamma_0^tS^*\left(T-t\right)\left(\alpha I+\Gamma_0^T\right)^{-1}\left(\mathbf{E}h-S(T)\left(\phi\left(0\right)+g\left(0,\phi\left(0\right)\right)\right)+g\left(T,Z\left(T\right)\right)\right)\\&-\int_0^t\left[I-\Gamma_s^tS^*\left(T-t\right)\left(\alpha I+\Gamma_s^T\right)^{-1}S(T-t)\right]AS\left(t-s\right)g\left(s,Z\left(s\right)\right)\mathrm{d}s\\&+\int_0^t\left[I-\Gamma_s^tS^*\left(T-t\right)\left(\alpha I+\Gamma_s^T\right)^{-1}S(T-t)\right]S\left(t-s\right)f\left(s,Z\left(\rho\left(s\right)\right)\right)\mathrm{d}s\\&+\int_0^t\left[I-\Gamma_s^tS^*\left(T-t\right)\left(\alpha I+\Gamma_s^T\right)^{-1}S(T-t)\right]S\left(t-s\right)\sigma\left(s,Z\left(\rho\left(s\right)\right)\right)\mathrm{d}W\left(s\right)\\&+\int_0^t\Gamma_s^tS^*\left(T-t\right)\left(\alpha I+\Gamma_s^T\right)^{-1}\varphi\left(s\right)\mathrm{d}W\left(s\right),\end{aligned}\tag{5.2}$$

The equation (5.2) is naturally obtained by inserting the control (5.1) into the mild solution of (1.1).

Let Z^{α} be a solution of (5.2). Writing this equation at $t = T$ yields

$$\begin{aligned}
Z^{\alpha}(T) = h - \alpha\left(\alpha I + \Gamma_0^T\right)^{-1}&\left(\mathbf{E}h - S(T)\left(\phi(0) + g(0,\phi(0))\right)\right.\\
&+ (-A)^{-\beta}(-A)^{\beta}\Big) g\left(T, Z^{\alpha}(T)\right)\\
&-\alpha\int_0^T \left(\alpha I + \Gamma_s^T\right)^{-1}(-A)^{1-\beta} S(T-s)(-A)^{\beta} g\left(s, Z^{\alpha}(s)\right)\mathrm{d}s\\
&-\alpha\int_0^T \left(\alpha I + \Gamma_s^T\right)^{-1} S(T-s) f\left(s, Z^{\alpha}(\rho(s))\right)\mathrm{d}s\\
&-\alpha\int_0^T \left(\alpha I + \Gamma_s^T\right)^{-1}\left[S(T-s)\sigma\left(s, Z^{\alpha}(\rho(s))\right) - \varphi(\tau)\right]dw(\tau)\Big).
\end{aligned} \tag{5.3}$$

By our assumption

$$\|f(s, Z^{\alpha}(\rho(s)))\|^p + \|\sigma(s, Z^{\alpha}(\rho(s)))\|_Q^p + \left\|(-A)^{\beta} g(s, Z^{\alpha}(s))\right\|^p \leq N_1$$

in $[0,T]\times\Omega$. Then there is a subsequence, still denoted by

$$\left\{f(s, Z^{\alpha}(\rho(s))), \sigma(s, Z^{\alpha}(\rho(s))), (-A)^{\beta} g(s, Z^{\alpha}(s))\right\},$$

weakly converging to, say, $(f(s,\omega), \sigma(s,\omega), g(s,\omega))$ in $H \times L_2^0 \times H$. The compactness of $S(t), t > 0$ implies that

$$\begin{cases}
S(T-s) f(s, Z^{\alpha}(\rho(s))) \to S(T-s) f(s),\\
S(T-s)\sigma(s, Z^{\alpha}(\rho(s))) \to S(T-s)\sigma(s)\\
S(t)(-A)^{\beta} g(s, Z^{\alpha}(s)) \to S(t) g(s) \ \ in \ \ [0,T]\times\Omega.
\end{cases} \tag{5.4}$$

Using the formula (5.3) one can show that $Z^{\alpha}(T)$ is a bounded sequence and so we may assume that $Z^{\alpha}(T) \to z^*$ weakly in H. On the other hand by the assumption (A9), for all $0 \leq \tau < T$

$$\alpha\left(\alpha I + \Gamma_s^T\right)^{-1} \to 0 \text{ strongly as } \alpha \to 0^+, \tag{5.5}$$

and moreover

$$\left\|\alpha\left(\alpha I + \Gamma_s^T\right)^{-1}\right\| \leq 1. \tag{5.6}$$

Thus from (5.3)-(5.6) by the Lebesgue dominated convergence theorem it follows that

$$\begin{aligned}
\mathbf{E}\|Z^{\alpha}(T) - h\|^p \leq 9^{p-1}&\left\|\alpha(\alpha + \Gamma_0^T)^{-1}\left(\mathbf{E}h - S(T)\left(\phi(0) + g(0,\phi(0))\right)\right)\right\|^p\\
&+9^{p-1}\mathbf{E}\left\|\alpha(\alpha + \Gamma_0^T)^{-1} g(T, Z^{\alpha}(T))\right\|^p\\
&+9^{p-1}\mathbf{E}\left(\int_0^T \left\|\alpha(\alpha + \Gamma_s^T)^{-1}\right\|\right.\\
&\left.\left\|(-A)^{1-\beta} S(T-s-\delta) S(\delta)\left[(-A)^{\beta} g(s, Z^{\alpha}(s)) - g(s)\right]\right\| ds\right)^p
\end{aligned}$$

$$+9^{p-1}\mathbf{E}\left(\int_0^T \left\|\alpha(\alpha+\Gamma_s^T)^{-1}AS(T-s)g\left(s\right)\right\| ds\right)^p$$
$$+9^{p-1}\mathbf{E}\left(\int_0^T \left\|\alpha(\alpha+\Gamma_s^T)^{-1}\right\| \left\|S(T-s)\left[f\left(s,Z^{\alpha}\left(\rho\left(s\right)\right)\right)-f\left(s\right)\right]\right\| ds\right)^p$$
$$+9^{p-1}\mathbf{E}\left(\int_0^T \left\|\alpha(\alpha+\Gamma_s^T)^{-1}S(T-s)f\left(s\right)\right\| ds\right)^p$$
$$+9^{p-1}\mathbf{E}\left(\int_0^T \left\|\alpha(\alpha+\Gamma_s^T)^{-1}\right\|^2 \left\|S(T-s)\left[\sigma\left(s,Z^{\alpha}\left(\rho\left(s\right)\right)\right)-\sigma\left(s\right)\right]\right\|_Q^2 ds\right)^{p/2}$$
$$+9^{p-1}\mathbf{E}\left(\int_0^T \left\|\alpha(\alpha+\Gamma_s^T)^{-1}S(T-s)\sigma\left(s\right)\right\|_Q^2 ds\right)^{p/2}$$
$$+9^{p-1}\mathbf{E}\left(\int_0^T \left\|\alpha(\alpha+\Gamma_s^T)^{-1}\varphi\left(s\right)\right\|_Q^2 ds\right)^{p/2}$$
$$\to 0 \ \text{ as } \ \alpha\to 0^{+}.$$

This gives the approximate controllability. Theorem is proved.

6. APPLICATIONS

In this section, we illustrate the obtained result. Let $H = L_2\left[0,\pi\right]$ and A be defined as follows

$$Az = z''$$

with domain

$$D\left(A\right) = \left\{z\left(\cdot\right)\in L_2\left[0,\pi\right] : z''\in L_2\left[0,\pi\right],\ z\left(0\right)=z\left(\pi\right)=0\right\}.$$

Recall that A is the infinitesimal generator of a strongly continuous semigroup $S\left(t\right), t>0$, on H which is analytic and self-adjoint, the eigenvalues are $-n^2$, $n\in N$, with corresponding normalized eigenvectors $e_n\left(\xi\right) := \left(2/\pi\right)^{1/2}\sin\left(n\xi\right)$. Moreover the following hold :

(a) $\left\{e_n : n\in N\right\}$ is an orthonormal basis of X.

(b) If $z\in D\left(A\right)$ then $A\left(z\right) = -\sum_{n=1}^{\infty} n^2\left\langle z,e_n\right\rangle e_n$.

(c) For $z\in H$, $\left(-A\right)^{-1/2}z = \sum_{n=1}^{\infty}\frac{1}{n}\left\langle z,e_n\right\rangle e_n$.

(d) The operator $\left(-A\right)^{1/2}$ is given as $\left(-A\right)^{1/2}z = \sum_{n=1}^{\infty} n\left\langle z,e_n\right\rangle e_n$ on the space $D\left[\left(-A\right)^{1/2}\right] = \left\{z\in H : \sum_{n=1}^{\infty} n\left\langle z,e_n\right\rangle e_n\in H\right\}$.

Consider the neutral system

$$\text{(6.1)}\quad \mathrm{d}\left[x\left(t,\xi\right)+\int_0^{\pi} b\left(\theta,\xi\right)r\left(t,x\left(t,\theta\right)\right)\mathrm{d}\theta\right] = \left[\frac{\partial^2}{\partial\xi^2}x\left(t,\xi\right)+p\left(t,x\left(t-h,\xi\right)\right)\right]\mathrm{d}t$$
$$+q\left(t,x\left(t-h,\xi\right)\right)\mathrm{d}\beta\left(t\right),$$

$$\text{(6.2)}\qquad x\left(t,0\right)=x\left(t,\pi\right)=0,\ t\geq 0,$$

(6.3) $$x(s,\xi)=\varphi(\xi)\in L_2[0,\pi],\ 0\le\xi\le\pi,\quad -h\le s\le 0,\quad h>0,$$

where $(p,q,r):[0,b]\times R\to R^3$ is a continuous function, $\beta(t)$ is a Brownian motion.

To write the initial-boundary value problem $(6.1)-(6.3)$ in the abstract form we assume the following.

(H1) The function b is measurable and

$$\int_0^\pi\int_0^\pi b^2(\theta,\xi)\,\mathrm{d}\theta\mathrm{d}\xi<\infty.$$

(H2) The function $(\partial/\partial\xi)\,b(\theta,\xi)$ is measurable, $b(\theta,0)=b(\theta,\pi)=0$, and let

$$L_1=\left[\int_0^\pi\int_0^\pi\left(\frac{\partial}{\partial\xi}b(\theta,\xi)\right)^2\mathrm{d}\theta\mathrm{d}\xi\right]^{1/2}<1.$$

Define $f,g,\sigma:[0,b]\times H\to H$ by

$$g(t,x)(\xi)=B(x)(\xi)=\int_0^\pi b(\theta,\xi)\,r(t,x(t,\theta))\,\mathrm{d}\theta,$$
$$f(t,x(\rho(t)))(\xi)=p(t,x(t-h,\xi)),\quad \sigma(t,x(\rho(t)))(\xi)=q(t,x(t-h,\xi))$$

From (H1) it is clear that B is bounded linear operator on H. Furthermore, $B(z)\in D\left[(-A)^{1/2}\right]$, and $\left\|(-A)^{1/2}B\right\|\le L_1$. In fact from the definition of B and (H2) it follows that

$$\begin{aligned}\langle B(x),e_n\rangle&=\int_0^\pi\left[\int_0^\pi b(\theta,\xi)\,x(\theta)\,\mathrm{d}\theta\right]e_n(\xi)\,\mathrm{d}\xi\\&=\frac{1}{n}\left(\frac{2}{\pi}\right)^{1/2}\left\langle\int_0^\pi\frac{\partial}{\partial\xi}b(\theta,\xi)\,x(\theta)\,\mathrm{d}\theta,\cos(n\xi)\right\rangle\\&=\frac{1}{n}\left(\frac{2}{\pi}\right)^{1/2}\langle B_1(x),\cos(n\xi)\rangle,\end{aligned}$$

where $B_1(x)=\int_0^\pi\frac{\partial}{\partial\xi}b(\theta,\xi)\,x(\theta)\mathrm{d}\theta$. From (H2) we know that $B_1:H\to H$ is a bounded linear operator with $\|B_1\|\le L_1$. Hence $\left\|(-A)^{1/2}B(x)\right\|=\|B_1(x)\|$, which implies the assertion.

(H3) The functions f and σ satisfy the conditions (C1) and (C2).

Thus the problem $(6.1)-(6.3)$ can be written in the abstract form

$$\mathrm{d}(x(t)+g(t,x(t)))=[Ax(t)+f(t,x(\rho(t)))]\,\mathrm{d}t+\sigma(t,x(\rho(t)))\,\mathrm{d}\beta(t),$$
$$x(0)=x_0,\quad t\in[0,T],$$

and all the conditions of Corollary 3.3 are satisfied. Thus by Corollary 3.3 the problem (6.1)-(6.3) has a unique mild solution.

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