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APPROXIMATION AND RAPID CONVERGENCE OF SOLUTIONS FOR PERIODIC NONLINEAR PROBLEMS

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ABSTRACT. We study existence and approximation of solutions of some second order nonlinear periodic boundary value problem of the type

$$-x''(t) = f(t, x, x'), \quad t \in [0, T],$$
$$x(0) = x(T), \quad x'(0) = x'(T),$$

in the presence of lower and upper solutions. We develop the upper and lower solutions method and the quasilinearization technique for the existence and approximation of solutions. We apply our theoretical results to a medical problem.

Key Words: Upper and Lower solutions, Quasilinearization, Biomathematical model

AMS (MOS) Subject Classification: 34A45, 34C25 , 92C50

1. INTRODUCTION

In this paper we study a nonlinear second order ordinary differential equation with periodic boundary conditions. We show the validity of the classical upper and lower solution method and of the monotone iterative technique [6] and present a new version related to [16]. This provides estimates for the solution and a numerical procedure to approximate the solution. Then we develop the quasilinearization technique [7] to obtain monotone sequences of approximate solutions converging quadratically to a solution. We improve previous results where the nonlinearity did not depend on the derivative [5, 9, 10]. To show the applicability of our techniques we apply the theoretical results to a medical problem: a model of blood flow inside an intracranial aneurysm [11, 12, 13].

We consider the nonlinear periodic boundary value problem (PBVP)

$$\begin{aligned} -x''(t) &= f(t,x,x'), \quad t \in [0,T], \\ x(0) &= x(T), \\ quadx'(0) &= x'(T), \end{aligned} \tag{1.1}$$

where $f \in C([0,T] \times \mathbb{R} \times \mathbb{R})$. We know that the linear homogeneous problem

$$\begin{aligned} -x''(t) + \lambda x(t) = 0, \quad t \in [0,T], \\ x(0) = x(T), \quad x'(0) = x'(T), \end{aligned}$$

has only a trivial solution if $\lambda \neq -\frac{4n^2\pi^2}{T^2}$, $n \in \mathbb{Z}$. Consequently, for $\lambda \neq -\frac{4n^2\pi^2}{T^2}$ and any $\sigma \in C[0,T]$, the nonhomogeneous problem

$$\begin{aligned} -x''(t) + \lambda x(t) = \sigma(t), \quad t \in [0,T], \\ x(0) = x(T), \quad x'(0) = x'(T), \end{aligned} \tag{1.2}$$

has a unique solution

$$x(t) = \int_0^T G_\lambda(t,s)\sigma(s)ds,$$

where $G_\lambda(t,s)$ is the Green's function, and for $\lambda > 0$,

$$G_\lambda(t,s) = \frac{1}{2\sqrt{\lambda}\sinh\sqrt{\lambda}\frac{T}{2}} \begin{cases} \cosh\sqrt{\lambda}(\frac{T}{2} + (t-s)), & \text{if } 0 \leq t < s \leq T \\ \cosh\sqrt{\lambda}(\frac{T}{2} + (s-t)), & \text{if } 0 \leq s < t \leq T, \end{cases}$$

and for $\lambda < 0$,

$$G_\lambda(t,s) = \frac{-1}{2\sqrt{|\lambda|}\sin\sqrt{|\lambda|}\frac{T}{2}} \begin{cases} \cos\sqrt{|\lambda|}(\frac{T}{2} + (t-s)), & \text{if } 0 \leq t < s \leq T \\ \cos\sqrt{|\lambda|}(\frac{T}{2} + (s-t)), & \text{if } 0 \leq s < t \leq T. \end{cases}$$

We note that if $\lambda > 0$, then $G_\lambda(t,s) > 0$ and if $\frac{-\pi^2}{T^2} \leq \lambda < 0$, then $G_\lambda(t,s) < 0$ on $(0,T) \times (0,T)$. Thus, we have the following maximum and anti-maximum principles.

Maximum principle 1.1. *If $\lambda > 0$, $\sigma \geq 0$ on $[0,T]$, then the solution x of* (1.2) *is such that $x \geq 0$ on $[0,T]$.*

Anti-maximum principle 1.2. *If $\frac{-\pi^2}{T^2} \leq \lambda < 0$ and $\sigma \geq 0$ on $[0,T]$, then the solution x of* (1.2) *is such that $x \leq 0$ on $[0,T]$. On the other hand, if $\sigma \leq 0$ on $[0,T]$, then $x \geq 0$ on $[0,T]$.*

2. UPPER AND LOWER SOLUTIONS METHOD

In this section, we study existence results of the BVP (1.1), using the method of upper and lower solutions. We show that in the presence of lower and upper solutions, there exists a unique solution of the PBVP (1.1). We recall the concept of lower and upper solution for the PBVP (1.1).

Definition 2.1. Let $\alpha \in C^2[0,T]$. We say that α is a lower solution of (1.1) if

$$-\alpha''(t) \leq f(t, \alpha(t), \alpha'(t)), \quad t \in [0,T]$$
$$\alpha(0) = \alpha(T),\ \alpha'(0) \geq \alpha'(T).$$

An upper solution β of the PBVP (1.1) is defined similarly by reversing the inequalities.

Definition 2.2. A continuous function $\omega : [0,\infty) \to (0,\infty)$, is called a Nagumo function if

$$\int_0^\infty \frac{sds}{\omega(s)} = +\infty.$$

We say that $f : I \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ satisfies a Nagumo condition on I relative to α, β, if there exists a Nagumo function ω such that

$$|f(t,x,y)| \leq \omega(|y|) \text{ on } I \times [\min \alpha, \max \beta] \times \mathbb{R}. \tag{2.1}$$

Now, we study existence results in the form of the following theorems.

Theorem 2.3. *Assume that α, β are lower and upper solutions of the PBVP (1.1). If $f \in C([0,T] \times \mathbb{R} \times \mathbb{R})$ and is strictly decreasing in x for each $(t,x') \in [0,T] \times \mathbb{R}$, then $\alpha(t) \leq \beta(t)$ for every $t \in [0,T]$.*

Proof. Define $w(t) = \alpha(t) - \beta(t)$, $t \in [0,T]$. Using the boundary conditions, we obtain

$$w(0) = w(T), \tag{2.2}$$

$$w'(0) \geq w'(T). \tag{2.3}$$

We claim that $w(t) \leq 0$ for every $t \in [0,T]$. If not, then $w(t)$ has a positive maximum at some $t_0 \in [0,T]$. If $t_0 = 0$ or T, then $w(0) = w(T)$ is a positive maximum so that

$$w(0) > 0,\ w'(0) \leq 0 \text{ and } w(T) > 0,\ w'(T) \geq 0. \tag{2.4}$$

The boundary conditions (2.3) and (2.4) imply that

$$w'(0) = 0,\ w'(T) = 0. \tag{2.5}$$

Now, using (2.5) and the decreasing property of $f(t,x,x')$ in x, we obtain

$$w''(0) = \alpha''(0) - \beta''(0) \geq -f(0, \alpha(0), \alpha'(0)) + f(0, \beta(0), \alpha'(0)) > 0,$$

which implies that the function w' is strictly increasing in some interval $(0, \delta)$ and hence

$$w'(t) > w'(0) = 0,\ 0 < t < \delta.$$

This implies that w is strictly increasing on $(0, \delta)$ and hence $w(t) > w(0)$, a contradiction. Hence $t_0 \in (0, T)$. Then, $w(t_0) > 0$, $w'(t_0) = 0$ and $w''(t_0) \le 0$. The definition of upper and lower solutions and the decreasing property of the function f in x gives

$$-w''(t_0) = -\alpha''(t_0) + \beta''(t_0) \le f(t_0, \alpha(t_0), \alpha'(t_0)) - f(t_0, \beta(t_0), \alpha'(t_0)) < 0,$$

a contradiction. □

Corollary 2.4. *Under the conditions of Theorem 2.3, the (PBVP)* (1.1) *has at most one solution.*

Theorem 2.5. *Assume that α, $\beta \in C^2[0, T]$ are lower and upper solutions of* (1.1) *respectively such that $\alpha < \beta$ on $[0, T]$. If $f : [0, T] \times \mathbb{R}^2 \to \mathbb{R}$ is continuous and satisfies a Nagumo condition, then there exists a solution x of the boundary value problem* (1.1) *such that*

$$\alpha(t) \le x(t) \le \beta(t),\ t \in [0, T].$$

Proof. Let $r = \max_{t\in[0,T]} \beta(t) - \min_{t\in[0,T]} \alpha(t)$, then there exists $N > 0$, such that

$$\int_0^N \frac{sds}{\omega(s)} > r.$$

Choose $C \ge \max\{N, \|\alpha'\|, \|\beta'\|\}$ and define $q(y) = \max\{-C, \min\{y, C\}\}$. Then $q(y) = y$ for $|y| \le C$ and $\mathrm{sgn}(q(y)) = \mathrm{sgn}(y)$. Moreover,

$$\int_0^C \frac{sds}{\omega(q(s))} = \int_0^C \frac{sds}{\omega(s)} \ge \int_0^N \frac{sds}{\omega(s)} > r. \tag{2.6}$$

Let $n \in \mathbb{N}$ and consider the modified problem

$$\begin{aligned} -x''(t) &= f_n(t, x, x'),\ t \in [0, T], \\ x(0) &= x(T),\ x'(0) = x'(T), \end{aligned} \tag{2.7}$$

where,

$$f_n(t, x, x') = \begin{cases} f(t, \beta(t), \beta'(t)), & \text{if } x \ge \beta(t) + \frac{1}{n}, \\ f(t, \beta(t), q(x')) + \big[f(t, \beta(t), \beta'(t)) - \\ \quad f(t, \beta(t), q(x'))\big] n(x - \beta(t)), & \text{if } \beta(t) < x < \beta(t) + \frac{1}{n}, \\ f(t, x, q(x')), & \text{if } \alpha(t) \le x \le \beta(t), \\ f(t, \alpha(t), q(x')) - \big[f(t, \alpha(t), \alpha'(t)) - \\ \quad f(t, \alpha(t), q(x'))\big] n(x - \alpha(t)), & \text{if } \alpha(t) - \frac{1}{n} < x < \alpha(t), \\ f(t, \alpha(t), \alpha'(t)), & \text{if } x \le \alpha(t) - \frac{1}{n}. \end{cases}$$

We note that $f_n(t,x,x')$ is continuous and bounded on $[0,T]\times\mathbb{R}^2$. Moreover, any solution x of (2.7) which satisfies the relations $\alpha(t)\le x(t)\le\beta(t)$ and $|x'(t)|\le C$ on $[0,T]$, is a solution of (1.1). For $t\in[0,T]$ and $x\in\mathbb{R}$, define

$$p(\alpha,x,\beta)=\max\{\alpha(t),\,\min\{x,\beta(t)\}\}.$$

Consider the system

$$\begin{aligned}-x''(t)&=sf_n(t,x,x')+(1-s)(\rho_n(t,x)-\lambda x),\ t\in[0,T]\\ x(0)&=x(T),\ x'(0)=x'(T),\end{aligned}\tag{2.8}$$

where $s\in[0,1]$, $\lambda>0$ and

$$\rho_n(t,x)=\frac{1}{\beta(t)-\alpha(t)}[(p(\alpha(t),x,\beta(t))-\alpha(t))(f(t,\beta(t),\beta'(t))+\lambda(\beta(t)+1/n))+ \\ (\beta(t)-p(\alpha(t),x,\beta(t)))(f(t,\alpha(t),\alpha'(t))+\lambda(\alpha(t)-1/n))],\ t\in[0,T].$$

For $s=0$, the system reduces to

$$\begin{aligned}-x''(t)+\lambda x(t)&=\rho_n(t,x),\ \ t\in[0,T],\\ x(0)&=x(T),\ x'(0)=x'(T)\end{aligned}\tag{2.9}$$

and for $s=1$, it is (2.7). That is, (2.8) has a solution for $s=0$. Now, for $s\in[0,1]$, we claim that any solution x_n of (2.8) satisfies

$$\alpha(t)-\frac{1}{n}\le x_n(t)\le\beta(t)+\frac{1}{n},\ t\in[0,T].\tag{2.10}$$

Once this is shown we can apply Schauder's fixed point theorem to conclude that (2.7) has a solution. To verify (2.10), we set $v_n(t)=x_n(t)-\beta(t)-1/n$, $t\in[0,T]$. Then the boundary conditions imply that

$$v_n(0)=v_n(T)\ \text{and}\ v_n'(0)\ge v_n'(T).\tag{2.11}$$

Assume that $\max\{v_n(t):t\in[0,T]\}=v_n(t_0)>0$. If $t_0=0$ or T, then we have

$$v_n(0)>0,\ v_n'(0)\le0\ \text{and}\ v_n(T)>0,\ v_n'(T)\ge0.\tag{2.12}$$

From the boundary conditions (2.11) and (2.12), we obtain $v_n'(0)=0$ and $v_n'(T)=0$. There exists $t_1\in(0,T)$ such that $v_n(t)\ge0$, $v_n'(t)\le0$ on $[0,t_1]$. For every $t\in[0,t_1]$, we have

$$-v_n''(t)=-x_n''(t)+\beta''(t)\le sf(t,\beta(t),\beta'(t))+(1-s)\Big[f(t,\beta(t),\beta'(t)) \\ +\lambda(\beta(t)+1/n)-\lambda x_n(t)\Big]-f(t,\beta(t),\beta'(t))=-\lambda(1-s)v_n(t)<0.$$

This implies that $v_n'(t)$ is strictly increasing on $[0,t_1)$ and hence $v_n'(t)>v_n'(0)=0$ on $[0,t_1)$, a contradiction. It follows that $t_0\in(0,T)$ and hence $v_n(t_0)>0$, $v_n'(t_0)=$

0 and $v_n''(t_0) \le 0$. However,

$$-v_n''(t_0) = -x_n''(t_0) + \beta''(t_0) \le sf(t_0, \beta(t_0), \beta'(t_0)) + (1-s)\Big[f(t_0, \beta(t_0), \beta'(t_0)) + \lambda(\beta(t_0) + 1/n) - \lambda x_n(t_0)\Big] - f(t_0, \beta(t_0), \beta'(t_0)) = -\lambda(1-s)v_n(t_0) < 0,$$

again a contradiction. Hence $x_n(t) \le \beta(t) + \frac{1}{n}$, $t \in [0,T]$. Similarly, we can show that $x_n(t) \ge \alpha(t) - 1/n$, $t \in [0,T]$.

The sequence $\{x_n\}$ of solutions of (2.7) is bounded and equicontinuous in $C^1[0,T]$ since f_n are bounded independently of n. Hence the Arzelà-Ascoli theorem guarantees the existence of a subsequence converging in $C^1[0,T]$ to a function $x \in C^1[0,T]$. Since (2.10) holds for every $n \in \mathbb{N}$ and every $t \in [0,T]$, it follows that

$$\alpha(t) \le x(t) \le \beta(t),\ t \in [0,T].$$

It remains to show that $|x'(t)| \le C$ on $[0,T]$. The boundary condition, $x(0) = x(T)$ implies that there exists $t^* \in (0,T)$ such that $x'(t^*) = 0$. Suppose that there exists $t_0 \in [0,T]$ such that $x'(t_0) \ge C$. Let $[t^*, t_2] \subset [0,T]$ be the maximal interval containing t_0 such that $x'(t) \ge 0$ on $[t^*, t_2]$. Let $\max\{x'(t) : t \in [t^*, t_2]\} = x'(t^{**}) = \hat{C}$, then $t^{**} \ne t^*$ and $\hat{C} \ge C$. It follows that

$$\int_0^{\hat{C}} \frac{sds}{\omega(q(s))} \ge \int_0^{C} \frac{sds}{\omega(q(s))} > r. \tag{2.13}$$

Now, for each $t \in [t^*, t_2]$, since $x \in [\min \alpha(t), \max \beta(t)]$ and $x' \ge 0$, we have

$$|-x''(t)| = |f(t, x, q(x'))| \le \omega(q(x')).$$

It follows that

$$\frac{x'(t)|x''(t)|}{\omega(q(x'))} \le x'(t).$$

Integrating from t^* to t^{**}, we obtain

$$\int_0^{\hat{C}} \frac{sds}{\omega(q(s))} \le x(t^{**}) - x(t^*) < \max_{t\in[0,T]} \beta(t) - \min_{t\in[0,T]} \alpha(t) < r,$$

a contradiction. Similarly, we can show that $x'(t) > -C$, $t \in [0,T]$.
Hence $|x'(t)| < C$, $t \in [0,T]$. □

3. QUASILINEARIZATION TECHNIQUE

In this section, we approximate our problem by the method of quasilinearization. We prove that under suitable conditions on the function f, there exists a monotone sequence of solutions of linear problems which converges to a solution of the nonlinear problem (1.1) and that the rate of convergence is quadratic.

Theorem 3.1. *Assume that*

($\mathbf{A_1}$) *α and $\beta \in C^2[0,T]$ are lower and upper solutions of* (1.1) *such that $\alpha < \beta$ on* $[0,T]$.

($\mathbf{A_2}$) *$f \in C^2([0,T] \times \mathbb{R}^2)$ and satisfies $f_x(t,x,x') \le -\lambda$, for some $\lambda > 0$. Moreover, we assume that $H(f) \ge 0$ on $[0,T] \times [\min \alpha(t), \max \beta(t)] \times [-C,C]$, where*

$$H(f) = (x-y)^2 f_{xx}(t,z_1,z_2) + 2(x-y)(x'-y') f_{xx'}(t,z_1,z_2) + (x'-y')^2 f_{x'x'}(t,z_1,z_2)$$

is the quadratic form of f with z_1 between x, y, and z_2 lies between x' and y'.

($\mathbf{A_3}$) *For $(t,x) \in [0,T] \times [\min \alpha(t), \max \beta(t)]$, $f_{x'}(t,x,x')$ satisfies*

$$|f_{x'}(t,x,y_1) - f_{x'}(t,x,y_2)| \le L|y_1 - y_2|, \; y_1, \; y_2 \in \mathbb{R},$$
$$x' f_{x'}(t,x,x') \ge 0 \text{ for } |x'| \ge C,$$

where $L > 0$ and C is as defined in Theorem 2.5.

Then, there exists a monotone sequence $\{w_n\}$ of solutions of linear problems converging uniformly and quadratically to a solution of the problem (1.1).

Proof. Let

$$S = \{(t,x,x') \in [0,T] \times \mathbb{R}^2 : (t,x,x') \in [0,T] \times [\min \alpha(t), \max \beta(t)] \times \mathbb{R}\}$$

and assume that

$$N = \max \{|f_{xx}(t,x,q(x'))|, \; |f_{xx'}(t,x,q(x'))|, \; |f_{x'x'}(t,x,q(x'))| : \; (t,x,x') \in S\}.$$

Then

$$|H(f)| \le N\|x-y\|_1^2 \text{ on } [0,T] \times [\min \alpha(t), \max \beta(t)] \times [-C,C], \tag{3.1}$$

where $\|x-y\|_1 = \|x-y\| + \|(x-y)'\|$ is the usual C^1 norm. Consider the boundary value problem

$$\begin{aligned} -x''(t) &= f(t,x,q(x')), \; t \in [0,T] \\ x(0) &= x(T), \; x'(0) = x'(T). \end{aligned} \tag{3.2}$$

We note that any solution $x \in C^2[0,T]$ of (3.2) with $\alpha(t) \le x \le \beta(t)$ is such that

$$|x'(t)| \le C \text{ on } [0,T]$$

and hence is a solution of (1.1). Therefore it suffices to study (3.2). Expanding $f(t,x,q(x'))$ about $(t,y,q(y')) \in S$ by Taylor's theorem and using (A_2), we have

$$f(t,x,q(x')) \ge f(t,y,q(y')) + f_x(t,y,q(y'))(x-y) + f_{x'}(t,y,q(y'))(q(x') - q(y')), \tag{3.3}$$

for $(t,x,x') \in S$. Define the function

$$F(t,x,x';y,y') = f(t,y,q(y')) + f_x(t,y,q(y'))(x-y) + f_{x'}(t,y,q(y'))[q(x') - q(y')], \tag{3.4}$$

where (t,x,x'), $(t,y,y') \in [0,T] \times \mathbb{R}^2$. Then F is continuous and bounded on S and therefore satisfies a Nagumo condition on $[0,T]$ relative to the pair α, β. Hence there exists a constant $C_1 > 0$ such that any solution x of the problem

$$-x''(t) + \lambda x(t) = F(t,x,x';y,y') + \lambda p(y,x,\beta),\ t \in [0,T],\ \lambda > 0$$
$$x(0) = x(T),\ x'(0) = x'(T),$$

with $\alpha(t) \le x \le \beta(t)$ satisfies $|x'(t)| \le C_1$ on $[0,T]$, where

$$p(y,x,\beta) = \max\{y,\ \min\{x,\ \beta(t)\}\}.$$

Moreover, $F_x = f_x(t,y,q(y')) \le -\lambda < 0$ and we have the following relations

$$\begin{cases} f(t,x,q(x')) \ge F(t,x,x';y,y') \\ f(t,x,q(x')) = F(t,x,x';x,x'), \end{cases} \tag{3.5}$$

for (t,x,x'), $(t,y,y') \in S$.

Now, we set $w_0 = \alpha$ and consider the linear problem

$$\begin{aligned} -x''(t) + \lambda x(t) &= F(t,x,x';w_0,w_0') + \lambda p(w_0,x,\beta),\ t \in [0,T],\ \lambda > 0 \\ x(0) &= x(T),\ x'(0) = x'(T). \end{aligned} \tag{3.6}$$

This is equivalent to the integral equation

$$x(t) = \int_0^1 G_\lambda(t,s)\big[F(s,x,x';w_0,w_0') + \lambda p(w_0,x,\beta)\big]ds.$$

Since $F(t,x,x';w_0,w_0') + \lambda p(w_0,x,\beta)$ is continuous and bounded on S, this integral equation has a fixed point (using again Schauder's fixed point theorem). Now, using (A_1) and (3.5), we obtain

$$\begin{aligned} -w_0''(t) + \lambda w_0(t) &\le f(t,w_0(t),w_0'(t)) + \lambda w_0(t) \\ &= F(t,w_0(t),w_0'(t);w_0(t),w_0'(t)) + \lambda p(w_0(t),w_0(t),\beta(t)),\ t \in [0,T], \end{aligned}$$

$$\begin{aligned} -\beta''(t) + \lambda\beta(t) &\ge f(t,\beta(t),\beta'(t)) + \lambda\beta(t) \\ &\ge F(t,\beta(t),\beta'(t);w_0(t),w_0'(t)) + \lambda p(w_0(t),\beta(t),\beta(t)),\ t \in [0,T], \end{aligned}$$

which imply that w_0 and β are lower and upper solution of (3.6). Hence, by Theorems 2.3, 2.5, there exists a unique solution w_1 of (3.6) such that $w_0(t) \le w_1(t) \le \beta(t)$, $|w_1'(t)| < C_1$, $t \in [0,T]$. In view of (3.5) and the fact that w_1 is a solution of (3.6), we have

$$-w_1''(t) = F(t,w_1(t),w_1'(t);w_0(t),w_0'(t)) \le f(t,w_1(t),q(w_1'(t))),\ t \in [0,T], \tag{3.7}$$

which implies that w_1 is a lower solution of (3.2). Now, consider the problem

$$\begin{aligned} -x''(t) + \lambda x(t) &= F(t,x,x';w_1,w_1') + \lambda p(w_1,x,\beta),\ t \in [0,T] \\ x(0) &= x(T),\ x'(0) = x'(T). \end{aligned} \tag{3.8}$$

In view of (A_1), (3.5) and (3.7), we can show that w_1 and β are lower and upper solutions of (3.8) and hence by Theorems 2.3, 2.5, there exists a unique solution w_2 of (3.8) such that $w_1(t) \leq w_2(t) \leq \beta(t)$, $|w_2'(t)| < C_1$, $t \in [0,T]$. Moreover w_2 is a lower solution of (1.1).

Continuing this process we obtain a monotone sequence $\{w_n\}$ of solutions satisfying

$$w_0 \leq w_1 \leq w_2 \leq w_3 \leq ... \leq w_{n-1} \leq w_n \leq \beta,\ t \in [0,T]$$

That is,

$$\alpha(t) \leq w_n(t) \leq \beta(t),\ |w_n'(t)| < C_1,\ n \in \mathbb{N},\ t \in [0,T], \tag{3.9}$$

where w_n is a solution of

$$\begin{gathered} -w_n''(t) = F(t, w_n, w_n'; w_{n-1}, w_{n-1}'),\ t \in [0,T] \\ w_n(0) = w_n(T),\ w_n'(0) = w_n'(T). \end{gathered} \tag{3.10}$$

Since $F(t, w_n, w_n'; w_{n-1}, w_{n-1}')$ is bounded, there exists $R > 0$ such that

$$|F(t, w_n, w_n'; w_{n-1}, w_{n-1}')| \leq R,\ n \in \mathbb{N},\ t \in [0,T].$$

Using the relation $w_n'(t) = w_n'(0) + \int_0^t w_n''(u)du$, we have

$$|w_n'(t) - w_n'(s)| \leq \int_s^t |F(u, w_n, w_n'; w_{n-1}, w_{n-1}')|du \leq R|t-s|, \tag{3.11}$$

for $t,\ s \in [0,T]$. From (3.9) and (3.11), it follows that the sequences

$$\{w_n^{(j)}(t)\}, (j = 0,1),\ n \in \mathbb{N},$$

are uniformly bounded and equicontinuous on $[0,T]$. The Arzelà- Ascoli theorem guarantees the existence of subsequences converging uniformly to $x^{(j)}(j = 0,1) \in C^1[0,T]$. Consequently, $F(t, w_n, w_n'; w_{n-1}, w_{n-1}') + \lambda p(w_{n-1}, w_n, \beta) \to f(t, x, q(x')) + \lambda x$ on $[0,T]$ as $n \to \infty$ which implies that x is a solution of (1.1).

Now, we show that the convergence is quadratic. For this, we set

$$v_n(t) = x(t) - w_n(t),\ t \in [0,T],\ n \in \mathbb{N},$$

where x is the solution of (1.1). Then, $v_n \in C^2[0,T]$, $v_n(t) \geq 0$, $n \in \mathbb{N}$, $t \in [0,T]$ and satisfies the boundary conditions

$$v_n(0) = v_n(T),\ v_n'(0) = v_n'(T).$$

The boundary condition $v_n(0) = v_n(T)$ implies the existence of $t_1 \in (0,T)$ such that $v_n'(t_1) = 0$. Now, in view of (3.4), we obtain

(3.12)

$$\begin{aligned} -v_n''(t) &= -x''(t) + w_n''(t) = f(t,x,x') - F(t, w_n, w_n'; w_{n-1}, w_{n-1}') \\ &= f_x(t, w_{n-1}, q(w_{n-1}'))v_n + f_{x'}(t, w_{n-1}, q(w_{n-1}')(x' - q(w_n')) + \frac{1}{2}|H(f)|, \end{aligned}$$

where

$$H(f) = (x - w_{n-1})^2 f_{xx}(t, c_1, c_2) + 2(x - w_{n-1})(x' - q(w'_{n-1}))f_{xx'}(t, c_1, c_2) \\ +(x' - q(w'_{n-1}))^2 f_{x'x'}(t, c_1, c_2),$$

$w_{n-1}(t) \le c_1 \le x(t)$ and c_2 lies between $q(w'_{n-1}(t))$ and $x'(t)$. Thus, $v_n(t)$ satisfies the problem

$$-v''_n(t) + \lambda v_n(t) = [f_x(t, w_{n-1}, q(w'_{n-1})) + \lambda]v_n(t) + f_{x'}(t, w_{n-1}, q(w'_{n-1}))(x' - q(w'_n)) \\ + \frac{1}{2}|H(f)|, \\ v_n(0) = v_n(T),\ v'_n(0) = v'_n(T).$$

This is equivalent to

$$0 \le v_n(t) = \int_0^T G_\lambda(t, s)\Big([f_x(s, w_{n-1}, q(w'_{n-1})) + \lambda]v_n(s) \\ + f_{x'}(s, w_{n-1}, q(w'_{n-1}))(x' - q(w'_n)) + \frac{1}{2}|H(f)|\Big)ds,$$

which in view of $(A_2)(f_x(t, x, x') + \lambda \le 0)$ and (3.1) implies that

(3.13)

$$v_n(t) \le \int_0^T G_\lambda(t, s)\big(f_{x'}(s, w_{n-1}, q(w'_{n-1}))(x' - q(w'_n)) + \frac{N}{2}\|v_{n-1}\|_1^2\big)ds \\ = \int_0^T G_\lambda(t, s)\big[f_{x'}(s, w_{n-1}, q(w'_{n-1}))v'_n + f_{x'}(s, w_{n-1}, q(w'_{n-1}))(w'_n - q(w'_n)) \\ + \frac{N}{2}\|v_{n-1}\|_1^2\big]ds.$$

Now, using (3.5), we have

(3.14)

$$-v''_n(t) = f(t, x, x') - F(t, w_n, w'_n; w_{n-1}, w'_{n-1}) \ge f(t, x, x') - f(t, w_n, q(w'_n)),\ t \in [0, T].$$

The condition $x' f_{x'}(t, x, x') \ge 0$ for $|x'| \ge C$, implies that

$$f(t, w_n(t), q(w'_n(t))) \le f(t, w_n(t), w'_n(t)),\ t \in [0, T],$$

so that (3.14) can be rewritten as

(3.15)

$$-v''_n(t) \ge f(t, x, x') - f(t, w_n(t), w'_n(t)) = f_x(t, d_1, d_2)v_n(t) + f_{x'}(t, d_1, d_2)v'_n(t) \\ \ge f_x(t, d_1, d_2)v_{n-1}(t) + f_{x'}(t, d_1, d_2)v'_n(t),\ t \in [0, T],$$

where $w_n(t) \le d_1 \le x(t)$ and d_2 lies between $x'(t)$ and $w'_n(t)$. Let $\mu(t) = e^{\int_0^t f_{x'}(s, d_1, d_2)ds}$ and $-l_1 \le f_{x'}(t, d_1, d_2) \le L_1$ on $[0, T] \times [\min w_0(t), \max \beta(t)] \times [-C_1, C_1]$, where $l_1,\ L_1 > 0$. Then

$$e^{-l_1 t} \le \mu(t) \le e^{L_1 t},\ t \in [0, T]. \tag{3.16}$$

Multiplying (3.15) by $\mu(t)$ and using (3.16), we obtain

$$\big(v_n'(t)\mu(t)\big)' \le \lambda\|v_{n-1}\|\mu(t) \le \lambda\|v_{n-1}\|e^{L_1 t},\ t\in[0,T], \tag{3.17}$$

where $\lambda = \max\{|f_x(t,x,x')| : t\in[0,T],\ x\in[\min w_0(t), \max\beta(t)],\ x'\in[-C_1,C_1]\}$. Thus,

$$\big(v_n'(t)\mu(t) - \lambda\|v_{n-1}\|\frac{e^{L_1 t}}{L_1}\big)' \le 0,\ t\in[0,T]. \tag{3.18}$$

This implies that the function $\psi(t) = v_n'(t)\mu(t) - \lambda\|v_{n-1}\|\frac{e^{L_1 t}}{L_1}$, is non-increasing in $t\in[0,T]$. Hence $\psi(0)\ge\psi(t_1)\ge\psi(T)$, which yields

$$v_n'(0) - \frac{\lambda}{L_1}\|v_{n-1}\| \ge -\frac{\lambda e^{L_1 t_1}}{L_1}\|v_{n-1}\| \ge v_n'(T)\mu_1(T) - \frac{\lambda e^{L_1 T}}{L_1}\|v_{n-1}\|.$$

Using the boundary conditions $v_n'(0) = v_n'(T)$, we obtain

$$v_n'(0) = v_n'(T) \le \frac{\lambda}{\mu(T)L_1}(e^{L_1 T} - e^{L_1 t_1})\|v_{n-1}\| \le \frac{\lambda}{\mu(T)L_1}(e^{L_1 T} - 1)\|v_{n-1}\|, \tag{3.19}$$

$$v_n'(T) = v_n'(0) \ge -\frac{\lambda}{L_1}(e^{L_1 t_1} - 1)\|v_{n-1}\| \ge -\frac{\lambda}{L_1}(e^{L_1 T} - 1)\|v_{n-1}\|. \tag{3.20}$$

Now the relation $\psi(0)\ge\psi(t)$, $t\in[0,T]$, together with (3.19), implies

$$v_n'(t) \le \frac{1}{\mu(t)}\Big[\frac{\lambda}{L_1}(\frac{e^{L_1 T}-1}{\mu(T)} - 1 + e^{L_1 t})\|v_{n-1}\|\Big] \le q_1\|v_{n-1}\|, \tag{3.21}$$

where $q_1 = \max\{\frac{\lambda}{L_1\mu(t)}(\frac{e^{L_1 T}-1}{\mu(T)} - 1 + e^{L_1 t}) : t\in[0,T]\}$. The relation $\psi(t)\ge\psi(T)$, $t\in[0,T]$, together with (3.20), implies

$$v_n'(t) \ge -\frac{\lambda}{\mu(t)L_1}\Big[\mu(T)(e^{L_1 T}-1) + e^{L_1 T} - e^{L_1 t}\Big]\|v_{n-1}\| \ge -q_2\|v_{n-1}\|, \tag{3.22}$$

where $q_2 = \max\{\frac{\lambda}{\mu(t)L_1}[\mu(T)(e^{L_1 T}-1) + e^{L_1 T} - e^{L_1 t}] : t\in[0,T]\}$. From (3.21) and (3.22), it follows that

$$|v_n'(t)| \le Q\|v_{n-1}\|,\ t\in[0,T], \tag{3.23}$$

where $Q = \max\{q_1, q_2\}$. We discuss three cases.

1. If for some $t\in[0,T]$, $w_n'(t) > C$, then

$$q(w_n'(t)) = C,\ 0 < w_n'(t) - q(w_n'(t)) \le w_n'(t) - x'(t)$$

and by (A_3), we obtain

$$\begin{aligned} f_{x'}(t, w_{n-1}(t), q(w_{n-1}'(t))) \le& f_{x'}(t, w_{n-1}(t), q(w_n'(t))) + L|q(w_{n-1}'(t)) - q(w_n'(t))| \\ &\le f_{x'}(t, w_{n-1}(t), q(w_n'(t))) + L(|v_n'(t)| + |v_{n-1}'(t)|). \end{aligned}$$

Hence using (3.23), we obtain

$$(w_n'(t)-q(w_n'(t)))f_{x'}(t,w_{n-1}(t),q(w_{n-1}'(t))) \le -v_n'(t)f_{x'}(t,w_{n-1}(t),q(w_n'(t)))+ L|v_n'(t)|(|v_n'(t)|+|v_{n-1}'(t)|)$$

$$\le -v_n'(t)f_{x'}(t,w_{n-1}(t),q(w_n'(t)))+LQ(1+Q)\|v_{n-1}\|_1^2.$$

2. If for some $t\in[0,T]$, $w_n'(t)<-C$, then

$$q(w_n'(t))=-C,\ 0>w_n'(t)-q(w_n'(t))\ge w_n'(t)-x'(t)$$

and by (A_3), we obtain

$$f_{x'}(t,w_{n-1}(t),q(w_{n-1}'(t))) \ge f_{x'}(t,w_{n-1}(t),q(w_n'(t)))-L|q(w_{n-1}'(t))-q(w_n'(t))|$$

$$\ge f_{x'}(t,w_{n-1}(t),q(w_n'(t)))-L(|v_n'(t)|+|v_{n-1}'(t)|),$$

hence

$$(w_n'(t)-q(w_n'(t)))f_{x'}(t,w_{n-1}(t),q(w_{n-1}'(t))) \le -v_n'(t)f_{x'}(t,w_{n-1}(t),q(w_n'(t)))+ LQ(1+Q)\|v_{n-1}\|_1^2.$$

3. If for some $t\in[0,T]$, $|w_n'(t)|\le C$, then $q(w_n'(t))=w_n'(t)$, $w_n'(t)-q(w_n'(t))=0$ and by the same process, we can show that

$$(w_n'(t)-q(w_n'(t)))f_{x'}(t,w_{n-1}(t),q(w_{n-1}'(t)))=0 \le -v_n'(t)f_{x'}(t,w_{n-1}(t),q(w_n'(t)))+ LQ(1+Q)\|v_{n-1}\|_1^2.$$

Thus, for every $t\in[0,T]$, we have

(3.24)

$$f_{x'}(t,w_{n-1},q(w_{n-1}'))(w_n'-q(w_n')) \le -v_n'(t)f_{x'}(t,w_{n-1}(t),q(w_n'(t)))+LQ(1+Q)\|v_{n-1}\|_1^2.$$

Using (3.24) in (3.13), we obtain

(3.25)

$$v_n(t) \le \int_0^T G_\lambda(t,s)\big[\big(f_{x'}(s,w_{n-1}(s),q(w_{n-1}'(s)))-f_{x'}(s,w_{n-1}(s),q(w_n'(s)))\big)v_n'(s) + (LQ(1+Q)+\frac{N}{2})\|v_{n-1}\|_1^2\big]ds$$

$$= \int_0^T G_\lambda(t,s)\big[\big(f_{x'}(s,w_{n-1}(s),q(w_{n-1}'(s)))-f_{x'}(s,w_{n-1}(s),q(w_n'(s)))\big)v_n'(s) + S_1\|v_{n-1}\|_1^2\big]ds,$$

where $S_1=LQ(1+Q)+\frac{N}{2}$.

Again, using (3.24) and (3.1) in (3.12), we obtain

(3.26)

$$-v_n''(t)\le \big(f_{x'}(t,w_{n-1},q(w_{n-1}')-f_{x'}(t,w_{n-1},q(w_n')\big)v_n'(t)+S_1\|v_{n-1}\|_1^2,\ t\in[0,T]$$

which implies that

(3.27)

$$v_n''(t)+\big(f_{x'}(t,w_{n-1},q(w_{n-1}')-f_{x'}(t,w_{n-1},q(w_n')\big)v_n'(t)\ge -S_1\|v_{n-1}\|_1^2,\ t\in[0,T].$$

Since $(t, w_{n-1}, q(w'_{n-1})) \in [0,T]\times[\min w_0(t), \max \beta(t)]\times[-C, C]$, and $f_{x'}$ is continuous, there exist L_2, $l_2 > 0$ such that

$$-l_2 \leq \big(f_{x'}(t, w_{n-1}, q(w'_{n-1}) - f_{x'}(t, w_{n-1}, q(w'_n))\big) \leq L_2,\ t \in [0,T].$$

Then the integrating factor $\mu_1(t) = e^{\int_0^t \left(f_{x'}(s,w_{n-1},q(w'_{n-1})-f_{x'}(s,w_{n-1},q(w'_n))\right)ds}$ satisfies

$$e^{-l_2 t} \leq \mu_1(t) \leq e^{L_2 t},\ t \in [0,T]. \tag{3.28}$$

Thus,

$$\big(v'_n(t)\mu_1(t)\big)' \geq -S_1 e^{L_2 t}\|v_{n-1}\|_1^2. \tag{3.29}$$

Integrating (3.29) from 0 to t_1, using (3.28) and the boundary conditions $v'_n(0) = v'_n(T)$, we obtain

$$v'_n(T) = v'_n(0) \leq \frac{S_1}{L_2}(e^{L_2 t_1} - 1)\|v_{n-1}\|_1^2 \leq \frac{S_1}{L_2}(e^{L_2 T} - 1)\|v_{n-1}\|_1^2. \tag{3.30}$$

Integrating (3.29) from t to T, using (3.28) and (3.30), we obtain

$$\begin{aligned} v'_n(t)\mu_1(t) &\leq v'_n(T)\mu_1(T) + \frac{4S_1}{L_2}(e^{L_2 T} - e^{L_2 t})\|v_{n-1}\|_1^2 \\ &\leq \frac{S_1}{L_2}\Big[\mu_1(T)(e^{L_2 T} - 1) + (e^{L_2 T} - e^{L_2 t})\Big]\|v_{n-1}\|_1^2, \end{aligned}$$

which implies that

(3.31)
$$v'_n(t) \leq \frac{S_1 e^{l_2 t}}{L_2}\Big[\mu_1(T)(e^{L_2 T} - 1) + (e^{L_2 T} - e^{L_2 t})\Big]\|v_{n-1}\|_1^2 \leq \delta_1\|v_{n-1}\|_1^2,\ t \in [0,T],$$

where

$$\delta_1 = \max_{[0,T]} \frac{S_1 e^{l_2 t}}{L_2}\Big[\mu_1(T)(e^{L_2 T} - 1) + (e^{L_2 T} - e^{L_2 t})\Big].$$

Again, integrating (3.29) from t_1 to T, using (3.28) and the boundary conditions $v'_n(0) = v'_n(T)$, we have

$$v'_n(0) = v'_n(T) \geq \frac{-S_1(e^{L_2 T} - 1)}{L_2\mu_1(T)}\|v_{n-1}\|_1^2. \tag{3.32}$$

If we integrate (3.29) from 0 to t, use (3.28) and (3.32), we obtain

$$v'_n(t)\mu_1(t) \geq \frac{-S_1}{L_2}\Big[\frac{(e^{L_2 T} - 1)}{\mu_1(T)} + (e^{L_2 t} - 1)\Big]\|v_{n-1}\|_1^2,\ t \in [0,T]$$

which implies that

$$v'_n(t) \geq \frac{-S_1 e^{l_2 t}}{L_2}\Big[\frac{(e^{L_2 T} - 1)}{\mu_1(T)} + (e^{L_2 t} - 1)\Big]\|v_{n-1}\|_1^2 \geq -\delta_2\|v_{n-1}\|_1^2,\ t \in [0,T], \tag{3.33}$$

where

$$\delta_2 = \max_{[0,T]} \frac{S_1 e^{l_2 t}}{L_2\mu_1(t)}\Big(\frac{(e^{L_2 T} - 1)}{(\mu_1(T) - 1)} + (e^{L_2 t} - 1)\Big).$$

From (3.31) and (3.33), it follows that

$$\|v'_n\| \leq \delta \|v_{n-1}\|_1^2, \quad \delta = \max\{\delta_1, \delta_2\}. \tag{3.34}$$

Now, using (3.31) in (3.25), we have

$$v_n(t) \leq \int_0^T G_\lambda(t,s)\big(\sigma\delta_1 + S_1\big)\|v_{n-1}\|_1^2 ds,$$

which implies that

$$\|v_n\| \leq \int_0^T G_\lambda(t,s)(\sigma\delta_1 + S_1)\|v_{n-1}\|_1^2 ds \leq D\|v_{n-1}\|_1^2, \tag{3.35}$$

where $\sigma = \max\{L_2, l_2\}$ and $D \geq (\sigma\delta_1 + S_1)\int_0^T G_\lambda(t,s)$. Let $R = \max\{\delta, D\}$, then (3.34) and (3.35) gives

$$\|v_n\|_1 \leq R\|v_{n-1}\|_1^2.$$

□

If $f_{x'} \equiv 0$, then it reduces to the case when the nonlinearity f is independent of the derivative x'. In this case the norm $\|.\|_1$ reduces to the norm $\|.\|$ and this case is studied in [5, 9]. Therefore we have extended previous results.

4. APPLICATION TO A BLOOD FLOW MODEL

Now, we apply our theoretical results to a medical problem, a biomathematical model of blood flow inside an intracranial aneurysm. An aneurysm is a local enlargement of the arterial lumen caused by congenital, traumatic, atherosclerotic or other factors. The natural history of the development of aneurysms consists of three phases: pathogenesis, enlargement and rupture. Aneurysmal subarachnoid hemorrhage is a major clinical problem in the world. The incidence of subarachnoid hemorrhage (SAH) is stable, at around six cases per 100000 patient a year [1]. The cause of SAH is a ruptured aneurysm in 85 percent of cases and SAH accounts for a quarter of cerebrovascular deaths [17] . The developments of the epidemiology and pathogenesis of intracranial aneurysms, methods of diagnosis, and approaches to treatment have been discussed by several authors [8, 14, 15]. Different sensitive, but non-invasive, imaging strategies for the diagnosis of intracranial aneurysms are now used [18]. For effective treatment of patients with intact incidental aneurysms, it is important to have adequate models in order to understand the evolution of aneurysms and to propose prognostic criteria upon which to make clinical recommendations. Mathematical models are now more relevant in biomedical practice [3] and several biomathematical models of intracranial aneurysms have been proposed in the literature [2, 11, 12] .
We consider the biomathematical model of blood flow inside an intracranial aneurysm

$$\begin{aligned} &x'' + px' + ax - bx^2 + cx^3 - F\cos(ht) = 0,\ t \in [0,T] \\ &x(0) = x(T),\ x'(0) = x'(T), \end{aligned} \tag{4.1}$$

where x represents the velocity of blood flow inside the aneurysm, and p, a, b, c, F, h are positive medical parameters depending on each patient. For example, F is related to the pulse pressure, h is the inverse of the cardiac frequency, p depends on the elastic properties of the aneurysm wall, ... for details, see [4, 11, 13].

Assume that $b^2 > \frac{16ac}{3}$, take $\lambda = \frac{a}{6} > 0$ and write the model (4.1) as

$$-x'' + \lambda x = f(t, x, x') = px' + \rho(x) - F\cos(ht), \tag{4.2}$$

where $\rho(x) = \frac{7a}{6}x - bx^2 + cx^3 = \upsilon(x) + \frac{a}{6}x$, $\upsilon(x) = ax - bx^2 + cx^3$. We note that the equation $\rho(x) = 0$ has three real roots namely, 0, x_1, x_2 with $(0 < x_1 < x_2)$, where $x_1 = \frac{b-(b^2-\frac{14ac}{3})^{\frac{1}{2}}}{2c}$, $x_2 = \frac{b+(b^2-\frac{14ac}{3})^{\frac{1}{2}}}{2c}$.

Clearly, $\rho(x) \geq 0$, for $x \in [0, x_1]$ and $\rho(x) \leq 0$, for $x \in [x_1, x_2]$. Let $\psi(x) = \rho(x) + \lambda x$, then

$$\psi_{\max} = \max \psi(x) = \psi(x_3) > 0,$$
$$\psi_{\min} = \min \psi(x) = \psi(x_4) < 0,$$

where $x_3 = \frac{b-(b^2-4ac)^{\frac{1}{2}}}{3c} \in (0, x_1)$ and $x_4 = \frac{b+(b^2-4ac)^{\frac{1}{2}}}{3c} \in (x_1, x_2)$. Moreover, $\psi_x(x) \leq 0$, for every $x \in [x_3, x_4]$ which implies that $\rho_x(x) \leq -\lambda$ for every $x \in [x_3, x_4]$. Similarly,

$$\upsilon(x) \geq 0 \text{ on } (0, \frac{3}{2}x_3),$$
$$\upsilon(x) \leq 0 \text{ on } (\frac{3}{2}x_3, \frac{3}{2}x_4).$$

Since $0 < x_3 < \frac{b-(b^2-4ac)^{\frac{1}{2}}}{2c}$, it follows that $\upsilon(x_3) > 0$ which implies that $\rho(x_3) - \lambda x_3 > 0$. Thus,

$$\psi_{\max} - 2\lambda(\frac{b-(b^2-4ac)^{\frac{1}{2}}}{3c}) > 0.$$

Let

$$F = \min\{\psi_{\max} - 2\lambda(\frac{b-(b^2-4ac)^{\frac{1}{2}}}{3c}), |\psi_{\min}| + \lambda(\frac{b+(b^2-4ac)^{\frac{1}{2}}}{3c})\}.$$

Taking $\alpha = x_3$ and $\beta = x_4$, we have $\alpha < \beta$ and

$$\alpha'' - \lambda\alpha + f(t, \alpha, \alpha') = -\lambda x_3 + \rho(x_3) - F\cos(ht) \geq \psi_{\max} - 2\lambda x_3 - F \geq 0,$$

$$\beta'' - \lambda\beta + f(t, \beta, \beta') = -\lambda x_4 + \rho(x_4) - F\cos(ht) \leq -2\lambda x_4 - |\psi_{\min}| + F \leq 0,$$

which imply that α, β are lower and upper solutions of (4.1).

Now, for $x \in [x_3, x_4]$, $t \in [0, T]$, we have

$$|f(t, x, x')| \leq p|x'| + K = \omega(|x'|),$$

where $K = \max\{\rho(x) - F\cos(ht) : t \in [0, T], x \in [x_3, x_4]\}$. Moreover,

$$\int_0^\infty \frac{sds}{\omega(s)} = \int_0^\infty \frac{sds}{ps + K} = \infty.$$

Thus the Nagumo condition is satisfied. Hence by Theorem 2.5, there exists a solution of (4.1) in $[x_3, x_4]$.

Now we approximate the solution of (4.1). The approximation scheme is given by solutions of the linear problems

$$-x'' + \lambda x = k(t, x, x'; y, y'),\ t \in [0, T],$$
$$x(0) = x(T),\ x'(0) = x'(T),$$

where

$$\begin{aligned} k(t, x, x'; y, y') &= f(t, y, y') + \rho_x(y)(x - y) + p(x' - y') \\ &= \rho(y) + \rho_x(y)(x - y) + px' - F\cos(ht). \end{aligned}$$

We rewrite

$$\begin{aligned} -x'' - px' + (\lambda - \rho_x(y))x &= \rho(y) - y\rho_x(y) - F\cos(ht) \\ x(0) = x(T),\ x'(0) &= x'(T), \end{aligned}$$

in the equivalent form

$$x(t) = \int_0^T G_\lambda(t, s)\big[\rho(y) - y\rho_x(y) - F\cos(hs)\big]ds.$$

Since $\lambda - \rho_x(y) > 0$ for $y \in [x_3, x_4]$, it follows that the Green's function $G_\lambda(t, s) > 0$ on $(0, T) \times (0, T)$.

The first approximation to the solution is x_3. Taking $w_0 = x_3$, the second approximation is given by

$$w_1(t) = \int_0^T G_\lambda(t, s)(\psi_{\max} - F\cos(hs))ds.$$

In general, the quasilinearization iteration scheme for the solution of (4.1) is given by

$$w_n = \int_0^T G_\lambda(t, s)\big(\rho(w_{n-1}) - w_{n-1}\rho_x(w_{n-1}) - F\cos(hs)\big)ds. \tag{4.3}$$

To show the sequence of iterates converges quadratically to the solution of the problem (4.1), we set $v_n(t) = x(t) - w_n(t)$. By the mean value theorem, (4.2) and (4.3), we obtain

$$\begin{aligned} -v_n''(t) + \lambda v_n(t) &= (-x''(t) + \lambda x(t)) \\ &\quad - (-w_n''(t) + \lambda w_n(t)) \\ &= px' + \rho(x) - [\rho(w_{n-1}) + \rho_x(w_{n-1})(w_n - w_{n-1}) + pw_n'] \\ &= pv_n' + \rho_x(w_{n-1})v_n + \frac{1}{2}\rho_{xx}(\xi)e_{n-1}^2 \\ &\le pv_n' + \rho_x(w_{n-1})v_n + d\|e_{n-1}\|^2, \end{aligned}$$

where d is a bound for $\frac{1}{2}|\rho_{xx}(x)|$. Thus, it follows that

$$-v_n''(t) - pv_n'(t) + (\lambda - \rho_x(w_{n-1})v_n(t) \le d\|e_{n-1}\|^2,$$

which implies that

$$\|v_n\| \le \delta\|v_{n-1}\|^2,$$

where $d\int_0^T |G_\lambda(t,s)|ds \leq \delta$. This shows that the iterates converges quadratically to a solution of the problem.

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