

# ASYMPTOTIC NONUNIFORM NONRESONANCE CONDITIONS FOR A NONLINEAR DISCRETE BOUNDARY VALUE PROBLEM

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**ABSTRACT.** Let  $\mathbb{T} := \{a+1, \dots, b+1\}$ . We study the solvability of nonlinear discrete two-point boundary value problem

$$\begin{cases} \Delta^2 u(t-1) + g(t, u(t)) = h(t), & t \in \mathbb{T}, \\ u(a) = u(b+2) = 0 \end{cases}$$

where  $h : \mathbb{T} \rightarrow \mathbb{R}$ ,  $g : \mathbb{T} \times \mathbb{R} \rightarrow \mathbb{R}$  satisfies

$$\alpha(t) \leq \liminf_{|x| \rightarrow \infty} x^{-1} g(t, x) \leq \limsup_{|x| \rightarrow \infty} x^{-1} g(t, x) \leq \beta(t)$$

uniformly on  $\mathbb{T}$ , and  $\alpha$  and  $\beta$  satisfy some nonresonance conditions of nonuniform type with respect to two consecutive eigenvalues of the associated linear problem. The proof is based on the Leray-Schauder continuation theorem.

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## 1. PRELIMINARIES

Let  $a, b \in \mathbb{N}$  with  $b - a > 2$ . Let  $\mathbb{T} := \{a + 1, \dots, b + 1\}$  and  $\hat{\mathbb{T}} := \{a, a + 1, \dots, b + 1, b + 2\}$ .

**Definition 1.1** Suppose that a function  $y : \hat{\mathbb{T}} \rightarrow \mathbb{R}$ . If  $y(t) = 0$  then  $t$  is a *zero* of  $y$ . If  $y(t) = 0$  and  $\Delta y(t) \neq 0$  then  $t$  is a *simple zero* of  $y$ . If  $y(t)y(t+1) < 0$  then we say that  $y$  has a *node* at the point  $s = \frac{ty(t+1) - (t+1)y(t)}{y(t+1) - y(t)} \in (t, t+1)$ . The nodes and simple zeros of  $y$  are called the *simple generalized zeros* of  $y$ .  $\square$

Let  $\mu$  be a real parameter. It is well-known that the linear eigenvalue problem

$$(1.1) \quad \begin{cases} \Delta^2 y(t-1) + \mu y(t) = 0, & t \in \mathbb{T}, \\ u(a) = u(b+2) = 0 \end{cases}$$

has exactly  $N := b - a + 1$  eigenvalues

$$(1.2) \quad \mu_1 < \mu_2 < \dots < \mu_N,$$

which are real and the eigenspace corresponding to any such eigenvalue is one dimensional. The following Lemma is crucial to the study of the nonlinear perturbations of

the linear problem (1.1). The required results are somewhat scattered in [3, Ch. 6~7], so we restate them here.

**Lemma 1.1**[3] Let  $(\mu_i, \psi_i), i \in \{1, \dots, N\}$ , denote eigenvalue pairs of (1.1) with

$$(1.3) \quad \sum_{t=a+1}^{b+1} \psi_j(t)\psi_j(t) = 1, \quad j \in \{1, \dots, N\}.$$

Then

(1)  $\psi_i$  has  $i - 1$  simple generalized zeros in  $[a + 1, b + 1]$ ; also if  $j \neq k$ , then

$$(1.4) \quad \sum_{t=a+1}^{b+1} \psi_j(t)\psi_k(t) = 0.$$

(2) if  $h : \{a + 1, \dots, b + 1\} \rightarrow \mathbb{R}$  is given, then the problem

$$(1.5) \quad \begin{cases} \Delta^2 u(t - 1) + \mu_k u(t) = h(t), & t \in \mathbb{T}, \\ u(a) = u(b + 2) = 0 \end{cases}$$

has solutions if and only if  $\sum_{t=a+1}^{b+1} h(t)\psi_k(t) = 0$ . □

In this paper, we study the existence of solutions of the nonlinear discrete boundary value problem

$$(1.6) \quad \begin{cases} \Delta^2 u(t - 1) + g(t, u(t)) = h(t), & t \in \mathbb{T}, \\ u(a) = u(b + 2) = 0 \end{cases}$$

where  $g : \mathbb{T} \times \mathbb{R} \rightarrow \mathbb{R}$  is a function.

**Definition 1.2** By a solution of (1.6) we mean a function  $u : \{a, a + 1, \dots, b + 1, b + 2\} \rightarrow \mathbb{R}$  which satisfies the difference equation and the boundary value conditions in (1.6). □

**Theorem 1.1** Assume

$$(1.7) \quad \alpha(t) \leq \liminf_{|x| \rightarrow \infty} x^{-1}g(t, x) \leq \limsup_{|x| \rightarrow \infty} x^{-1}g(t, x) \leq \beta(t)$$

uniformly on  $\mathbb{T}$ , where  $\alpha, \beta : \mathbb{T} \rightarrow \mathbb{R}$  are such that there exists a positive integer  $m \in \{1, \dots, N - 1\}$  with

$$(1.8) \quad \mu_m \leq \alpha(t) \leq \beta(t) \leq \mu_{m+1}, \quad t \in \mathbb{T},$$

$$(1.9) \quad \mu_m < \alpha(\tau_\alpha), \quad \text{for some } \tau_\alpha \in \mathbb{T} \setminus \{t \in \mathbb{T} \mid \psi_m(t) = 0\},$$

$$(1.10) \quad \beta(\tau_\beta) < \mu_{m+1}, \quad \text{for some } \tau_\beta \in \mathbb{T} \setminus \{t \in \mathbb{T} \mid \psi_{m+1}(t) = 0\}.$$

Then for each  $h : \mathbb{T} \rightarrow \mathbb{R}$ , (1.6) has at least one solution. □

**Theorem 1.2** Let  $g : \mathbb{T} \times \mathbb{R} \rightarrow \mathbb{R}$  be such that

$$\alpha(t) \leq \frac{g(t, u) - g(t, v)}{u - v} \leq \beta(t), \quad t \in \mathbb{T}, u \neq v$$

with  $\alpha$  and  $\beta$  as in Theorem 1.1. Then (1.6) has a unique solution.  $\square$

Conditions (1.8)-(1.10) are known as the ‘asymptotic nonuniform nonresonance condition’, whereas the stronger condition

$$\mu_m + \delta \leq \alpha(t) \leq \beta(t) \leq \mu_{m+1} - \delta, \quad t \in \mathbb{T},$$

(where  $\delta \in (0, \infty)$  is fixed), is called the ‘uniform nonresonance condition’. The analogue of Theorem 1.1 and 1.2 were obtained for periodic-Dirichlet problems of semilinear wave equations by Mawhin and Ward [5], and for boundary value problems of second order ordinary differential equations by Mawhin [6]. The main tool we use is the Leray-Schauder continuation theorem [7].

For results on the existence of solutions of discrete equations subject to diverse boundary conditions, see Agarwal and O’Regan [1-2], Kelley and Peterson [3], Rodriguez [8], Thompson and Tisdell [9] and the references therein. However none of these consider the problem under the ‘asymptotic nonuniform nonresonance’ condition, and no uniqueness results were proved.

## 2. EXISTENCE OF EXTREMAL SOLUTIONS

Let

$$(2.1) \quad D := \{(0, u(a+1), \dots, u(b+1), 0) \mid u(t) \in \mathbb{R}, t \in \mathbb{T}\}.$$

Then  $D$  is a Hilbert space under the inner product

$$\langle u, v \rangle = \sum_{t=a+1}^{b+1} u(t)v(t),$$

and the corresponding norm is

$$\|u\| := \sqrt{\langle u, u \rangle} = \left( \sum_{t=a+1}^{b+1} u(t)u(t) \right)^{1/2}.$$

Notice that  $D$  is also a Hilbert space under the inner product

$$\langle u, v \rangle_1 = \sum_{t=a}^{b+1} \Delta u(t) \Delta v(t),$$

and the corresponding norm is

$$\|u\|_1 := \sqrt{\langle u, u \rangle_1} = \left( \sum_{t=a}^{b+1} \Delta u(t) \Delta u(t) \right)^{1/2}.$$

For  $u \in D$ , let us write

$$(2.2) \quad u(t) = \bar{u}(t) + \tilde{u}(t)$$

where

$$(2.3) \quad \bar{u}(t) = \sum_{j=1}^m \langle u, \psi_j \rangle \psi_j(t), \quad \tilde{u}(t) = \sum_{j=m+1}^N \langle u, \psi_j \rangle \psi_j(t)$$

Obviously,  $D = \bar{D} \oplus \tilde{D}$  with

$$(2.4) \quad \bar{D} = \text{span}\{\psi_1, \dots, \psi_m\}, \quad \tilde{D} = \text{span}\{\psi_{m+1}, \dots, \psi_N\}.$$

**Lemma 2.1** ([4, Lemma 2.3]) Let  $u, w \in D$ . Then

$$\sum_{k=a+1}^{b+1} w(k) \Delta^2 u(k-1) = - \sum_{k=a}^{b+1} \Delta u(k) \Delta w(k).$$

**Lemma 2.2** Let  $\alpha : \mathbb{T} \rightarrow \mathbb{R}$  be a function such that

$$(2.5) \quad \mu_m \leq \alpha(t), \quad t \in \mathbb{T},$$

with

$$(2.6) \quad \mu_m < \alpha(\tau_\alpha), \quad \text{for some } \tau_\alpha \in \mathbb{T} \setminus \{t \in \mathbb{T} \mid \psi_m(t) = 0\}.$$

Then there is a positive number  $\delta_1 > 0$  such that for any  $p : \mathbb{T} \rightarrow \mathbb{R}$  satisfying

$$(2.7) \quad \alpha(t) \leq p(t), \quad t \in \mathbb{T}$$

and  $\bar{u} \in \bar{D}$ , we have

$$(2.8) \quad \sum_{t=a+1}^{b+1} (-\Delta^2 \bar{u}(t-1) - p(t)\bar{u}(t))\bar{u}(t) \leq -\delta_1 \sum_{a+1}^{b+1} [\bar{u}(t)]^2.$$

*Proof.* We divided the proof into two cases.

*Case 1.*  $m \geq 2$ .

By assumption (2.7), it is sufficient to prove that

$$(2.9) \quad \sum_{t=a+1}^{b+1} (-\Delta^2 \bar{u}(t-1) - \alpha(t)\bar{u}(t))\bar{u}(t) \leq -\delta_1 \sum_{t=a+1}^{b+1} [\bar{u}(t)]^2, \quad u \in \bar{D}.$$

Suppose on the contrary that there is no such  $\delta_1 > 0$ . Then there is a sequence  $\{\bar{u}_k\} \in \bar{D}$  with  $\|\bar{u}_k\| = 1$  and

$$(2.10) \quad -k^{-1} \leq \sum_{t=a+1}^{b+1} (-\Delta^2 \bar{u}_k(t-1) - \alpha(t)\bar{u}_k(t))\bar{u}_k(t), \quad k = 1, 2, \dots$$

For each  $k \in \mathbb{N}$ , we will write

$$(2.11) \quad \bar{u}_k = v_k + w_k$$

where

$$(2.12) \quad v_k \in \text{span}\{\psi_1, \dots, \psi_{m-1}\}, \quad w_k \in \text{span}\{\psi_m\}.$$

Now since  $\alpha(t) \geq \mu_m$  on  $\mathbb{T}$ , we have from (2.10) that

$$-k^{-1} \leq \sum_{t=a+1}^{b+1} (-\Delta^2 \bar{u}_k(t-1) - \mu_m \bar{u}_k(t)) \bar{u}_k(t), \quad k = 1, 2, \dots$$

which reduces to

$$(2.13) \quad \begin{aligned} -k^{-1} &\leq \sum_{t=a+1}^{b+1} (-\Delta^2 v_k(t-1) - \mu_m v_k(t)) v_k(t) \\ &\leq (\mu_{m-1} - \mu_m) \|v_k\|^2. \end{aligned}$$

so that  $\|v_k\| \rightarrow 0$  as  $k \rightarrow \infty$ .

Now since  $1 = \|\bar{u}_k\|^2 = \|v_k\|^2 + \|w_k\|^2$ , we have that a subsequence of  $\{\bar{u}_k\}$ , which we may relabel as  $\{\bar{u}_k\}$ , converges strongly to some  $w^* := \nu \psi_m$  with  $\nu \in \{+, -\}$ . Consequently,

$$(2.14) \quad \begin{aligned} -k^{-1} &\leq \sum_{t=a+1}^{b+1} (-\Delta^2 \bar{u}_k(t-1) - \alpha(t) \bar{u}_k(t)) \bar{u}_k(t) \\ &= \sum_{t=a+1}^{b+1} (-\Delta^2 v_k(t-1) - \alpha(t) v_k(t)) v_k(t) - 2 \sum_{t=a+1}^{b+1} \alpha(t) v_k(t) w_k(t) \\ &\quad + \sum_{t=a+1}^{b+1} (-\Delta^2 w_k(t-1) - \alpha(t) w_k(t)) w_k(t) \\ &\leq -(\mu_m - \mu_{m-1}) \|v_k\|^2 - 2 \sum_{t=a+1}^{b+1} \alpha(t) v_k(t) w_k(t) \\ &\quad + \sum_{t=a+1}^{b+1} (\mu_m - \alpha(t)) |w_k(t)|^2. \end{aligned}$$

Using  $v_k \rightarrow 0$  in  $\bar{D}$  and  $w_k \rightarrow w^*$  as  $k \rightarrow \infty$ , we obtain

$$(2.15) \quad -k^{-1} \leq \sum_{t=a+1}^{b+1} (\mu_m - \alpha(t)) |w^*(t)|^2 = \sum_{t=a+1}^{b+1} (\mu_m - \alpha(t)) |\psi_m(t)|^2,$$

and since  $\mu_m \leq \alpha(t)$  on  $\mathbb{T}$ , we have

$$(2.16) \quad \sum_{t=a+1}^{b+1} (\mu_m - \alpha(t)) |\psi_m(t)|^2 = 0.$$

However, (2.16) contradicts (2.6). This contradiction proves (2.9) and hence (2.8) hold.

*Case 2.  $m = 1$ .*

In this case, (2.11) reduces to

$$\bar{u}_k = v_k + w_k = 0 + w_k$$

with  $v_k \in \{0\}$  and  $w_k \in \text{span}\{\psi_1\}$ . It is easy to see that the argument used in Case 1 still works in this case. □

Using a similar method with obvious changes, we get the following

**Lemma 2.3** Let  $\beta : \mathbb{T} \rightarrow \mathbb{R}$  be a function such that

$$\beta(t) \leq \mu_{m+1}, \quad t \in \mathbb{T},$$

with

$$\beta(\tau_\alpha) < \mu_{m+1}, \quad \text{for some } \tau_\alpha \in \mathbb{T} \setminus \{t \in \mathbb{T} \mid \psi_{m+1}(t) = 0\}.$$

Then there is a positive number  $\delta_2 > 0$  such that for any  $p : \mathbb{T} \rightarrow \mathbb{R}$  satisfying

$$p(t) \leq \beta(t), \quad t \in \mathbb{T}$$

and  $\tilde{u} \in \tilde{D}$ , we have

$$\sum_{t=a+1}^{b+1} (-\Delta^2 \tilde{u}(t-1) - p(t)\tilde{u}(t))\tilde{u}(t) \geq \delta_2 \sum_{t=a+1}^{b+1} [\tilde{u}(t)]^2. \quad \square$$

**Lemma 2.4** Let  $\alpha, \beta : \mathbb{T} \rightarrow \mathbb{R}$  be functions such that

$$(2.17) \quad \mu_m \leq \alpha(t) \leq \beta(t) \leq \mu_{m+1}, \quad t \in \mathbb{T},$$

$$(2.18) \quad \mu_m < \alpha(\tau_\alpha), \quad \text{for some } \tau_\alpha \in \mathbb{T} \setminus \{t \in \mathbb{T} \mid \psi_m(t) = 0\},$$

$$(2.19) \quad \beta(\tau_\beta) < \mu_{m+1}, \quad \text{for some } \tau_\beta \in \mathbb{T} \setminus \{t \in \mathbb{T} \mid \psi_{m+1}(t) = 0\}.$$

Then there are  $\delta > 0$  and  $\epsilon > 0$  such that for any  $p : \mathbb{T} \rightarrow \mathbb{R}$  satisfying

$$(2.20) \quad \alpha(t) - \epsilon \leq p(t) \leq \beta(t) + \epsilon, \quad t \in \mathbb{T},$$

we have

$$(2.21) \quad \|(-\Delta^2 \rho - p)u(\cdot)\| \geq \delta \|u\|, \quad u \in D,$$

where

$$\rho u(t) := u(t-1), \quad \Delta^2 \rho u(a) := 0, \quad \Delta^2 \rho u(b+2) := 0,$$

(so that  $(-\Delta^2 \rho - p)u \in D$  for each  $u \in D$ ).

*Proof.* Suppose the conclusion of the lemma is false. Then there exists a sequence  $\{u_k\}$  in  $D$  with  $\|u_k\| = 1$  and a sequence  $\{p_k : \mathbb{T} \rightarrow \mathbb{R}\}$  with

$$(2.22) \quad \alpha(t) - \frac{1}{k} \leq p_k(t) \leq \beta(t) + \frac{1}{k}, \quad t \in \mathbb{T},$$

for  $k \in \mathbb{N}$ , and

$$(2.23) \quad \|(-\Delta^2 \rho - p)u_k\| \leq \frac{1}{k}, \quad k = 1, 2, \dots$$

That is

$$(2.24) \quad -\Delta^2 u_k(t-1) - p u_k(t) = f_k(t), \quad t \in \mathbb{T}$$

with  $f_k \in D$ ,  $\|f_k\| \leq k^{-1}$  and  $\|u_k\| = 1$  for  $k \in \mathbb{N}$ .

Write  $u_k = \bar{u}_k + \tilde{u}_k$ , where  $\bar{u}_k \in \bar{D}$  and  $\tilde{u}_k \in \tilde{D}$  for  $k \in \mathbb{N}$ . By (2.24), we have that

$$\sum_{t=a+1}^{b+1} (-\Delta^2 u_k(t-1) - p_k(t)u_k(t))(\tilde{u}_k(t) - \bar{u}_k(t)) = \sum_{t=a+1}^{b+1} f_k(t)(\tilde{u}_k(t) - \bar{u}_k(t))$$

which reduces upon expansion to

$$\begin{aligned} & \sum_{t=a+1}^{b+1} (-\Delta^2 \tilde{u}_k(t-1) - p_k(t)\tilde{u}_k(t))\tilde{u}_k(t) \\ (2.25) \quad & - \sum_{t=a+1}^{b+1} (-\Delta^2 \bar{u}_k(t-1) - p_k(t)\bar{u}_k(t))\bar{u}_k(t) \\ & = \sum_{t=a+1}^{b+1} f_k(t)(\tilde{u}_k(t) - \bar{u}_k(t)). \end{aligned}$$

Now by (2.22),

$$(2.26) \quad \alpha(t) \leq p_k(t) + \frac{1}{k}, \quad p_k(t) - \frac{1}{k} \leq \beta(t), \quad t \in \mathbb{T},$$

for  $k \in \mathbb{N}$ . So, using Lemma 2.2 and 2.3, we obtain the existence of  $\delta_1 > 0$  and  $\delta_2 > 0$  such that for all  $k \in \mathbb{N}$ , we have

$$\sum_{t=a+1}^{b+1} (-\Delta^2 \bar{u}_k(t-1) - (p_k(t) + \frac{1}{k})\bar{u}_k(t))\bar{u}_k(t) \leq -\delta_1 \|\bar{u}_k\|^2,$$

and

$$\sum_{t=a+1}^{b+1} (-\Delta^2 \tilde{u}_k(t-1) - (p_k(t) - \frac{1}{k})\tilde{u}_k(t))\tilde{u}_k(t) \geq \delta_2 \|\tilde{u}_k\|^2.$$

Combining with (2.25) and using Schwarz inequality, this gives

$$\delta_2 \|\tilde{u}_k\|^2 - \frac{1}{k} \|\tilde{u}_k\|^2 + \delta_1 \|\bar{u}_k\|^2 - \frac{1}{k} \|\bar{u}_k\|^2 \leq \|f_k\| \|\tilde{u}_k - \bar{u}_k\|,$$

and hence

$$\delta_2 \|\tilde{u}_k\|^2 + \delta_1 \|\bar{u}_k\|^2 \leq \frac{4}{k}$$

which implies that  $u_k = \bar{u}_k + \tilde{u}_k$  converges strongly to zero. This contradicts  $\|u_k\| = 1$  and thus proves the lemma. □

### 3. PROOF OF THE MAIN RESULTS

We now proceed to the proofs of Theorem 1.1 and 1.2 stated in the introduction.

*Proof of Theorem 1.1.* Let  $\delta > 0$  and  $\epsilon > 0$  be given by Lemma 2.4. By (1.7) we can find  $r > 0$  such that for  $t \in \mathbb{T}$  and all  $u$  with  $|u| \geq r$ , we have

$$(3.1) \quad \alpha(t) - \epsilon \leq u^{-1}g(t, u) \leq \beta(t) + \epsilon.$$

This implies that

$$|g(t, u)| \leq \max\{|\mu_{m+1}| + \epsilon, |\mu_m| + \epsilon\}|u| + h_r(t), \quad t \in \mathbb{T}, u \in \mathbb{R},$$

where  $h_r : \mathbb{T} \rightarrow \mathbb{R}$  is a fixed function. Consequently, the mapping  $G$  defined on  $D$  by

$$(Gu)(t) = g(t, u(t)), \quad t \in \mathbb{T}; (Gu)(a) = (Gu)(b + 2) = 0,$$

will map  $D$  continuously into itself and take bounded sets into bounded sets.

Using that fact that  $D$  is finite dimensional and according to the Leray-Schauder continuation theorem [6], we have that (1.6) have solutions if the set of possible solutions of the family of equations

$$(3.2) \quad \begin{cases} \Delta^2 u(t - 1) + (1 - \lambda) \frac{\alpha(t) + \beta(t)}{2} u(t) + \lambda g(t, u(t)) = \lambda h(t), \\ u(a) = u(b + 2) = 0 \end{cases}$$

( $t \in \mathbb{T}, \lambda \in [0, 1]$ ), is a priori bounded independently of  $\lambda$ . Define  $\Gamma$  on  $\mathbb{T} \times \mathbb{R}$  by

$$\Gamma(t, u) = \begin{cases} u^{-1}g(t, u), & |u| \geq r, \\ r^{-1}g(t, r)(u/r) + (1 - u/r) \frac{\alpha(t) + \beta(t)}{2}, & 0 \leq u \leq r, \\ r^{-1}g(t, -r)(u/r) + (1 + u/r) \frac{\alpha(t) + \beta(t)}{2}, & -r \leq u \leq 0, \end{cases}$$

and  $\sigma$  on  $\mathbb{T} \times \mathbb{R}$  by

$$\sigma(t, u) = g(t, u) - \Gamma(t, u)u.$$

It is easy to check that

$$(3.3) \quad \alpha(t) - \epsilon \leq \Gamma(t) \leq \beta(t) + \epsilon, \quad t \in \mathbb{T}, u \in \mathbb{R},$$

and there exists a constant  $M_1 \geq 0$ , such that

$$(3.4) \quad |\sigma(t, u)| \leq M_1.$$

If  $u : D \rightarrow \mathbb{R}$  is a solution of (3.2) for some  $\lambda \in [0, 1]$ , then

$$(3.5) \quad \begin{cases} \Delta^2 u(t - 1) + [(1 - \lambda) \frac{(\alpha(t) + \beta(t))}{2} + \lambda \Gamma(t, u(t))]u(t) \\ \hspace{15em} = \lambda[h(t) - \sigma(t, u(t))], \\ u(a) = u(b + 2) = 0. \end{cases}$$

( $t \in \mathbb{T}, \lambda \in [0, 1]$ ). Using (3.3) and the fact that

$$\alpha(t) \leq \frac{(\alpha(t) + \beta(t))}{2} \leq \beta(t), \quad t \in \mathbb{T},$$

it follows that

$$\alpha(t) - \epsilon \leq [(1 - \lambda) \frac{(\alpha(t) + \beta(t))}{2} + \lambda \Gamma(t, u(t))] \leq \beta(t) + \epsilon,$$



and hence using Lemma 2.4, (3.5) and (3.4), we obtain

$$\begin{aligned} M_1 + \|h\| &\geq \|\lambda[h(\cdot) - \sigma(\cdot, u(\cdot))]\| \\ &= \|\Delta^2 u(\cdot - 1) + [(1 - \lambda)\frac{\alpha(\cdot) + \beta(\cdot)}{2} + \lambda\Gamma(\cdot, u(\cdot))]u(\cdot)\| \\ &\geq \delta\|u\|, \end{aligned}$$

i.e.

$$\|u\| \leq \delta^{-1}(M_1 + \|h\|).$$

This completes the proof of Theorem 1.1. □

*Proof of Theorem 1.2.* Obviously, existence follows as in Theorem 1.1.

Now let  $u$  and  $v$  are solutions of (1.6), then, letting  $w = u - v$  will be a solution of

$$(3.6) \quad \begin{cases} \Delta^2 w(t - 1) + [g(t, u(t)) - g(t, v(t))] = 0, & t \in \mathbb{T}, \\ w(a) = w(b + 2) = 0. \end{cases}$$

Setting

$$\gamma(t, w(t)) = \begin{cases} w^{-1}[g(t, u(t)) - g(t, v(t))], & w \neq 0, \\ \frac{\alpha(t) + \beta(t)}{2}, & w = 0, \end{cases}$$

we see that (3.6) can be written

$$(3.7) \quad \begin{cases} \Delta^2 w(t - 1) + \gamma(t, w(t))w(t) = 0, & t \in \mathbb{T}, \\ u(a) = u(b + 2) = 0, \end{cases}$$

with

$$(3.8) \quad \alpha(t) \leq \gamma(t, w(t)) \leq \beta(t), \quad t \in \mathbb{T}.$$

Therefore, by Lemma 2.4, there exist  $\delta > 0$  and  $\bar{\epsilon} > 0$  such that for any  $p : \mathbb{T} \rightarrow \mathbb{R}$  satisfying

$$(3.9) \quad \bar{\alpha}(t) - \bar{\epsilon} \leq p(t) \leq \beta(t) + \bar{\epsilon}, \quad t \in \mathbb{T},$$

we have

$$(3.10) \quad \|(-\Delta^2 \rho - p)u(\cdot)\| \geq \delta\|u\|, \quad u \in D,$$

where  $\rho u(t) := u(t - 1)$ . This together with (3.8) and implies that

$$(3.11) \quad \|(-\Delta^2 \rho - \gamma(t, w(t)))w(\cdot)\| \geq \delta\|w\|.$$

Combining (3.11) with (3.7), it follows that

$$0 \geq \delta\|w\|,$$

and subsequently  $w = 0$ , i.e.  $u = v$ , and the proof is complete. □

#### 4. EXAMPLE AND REMARK

From [3, Example 4.1], we know that the linear eigenvalues and the eigenfunctions of the problem

$$(4.1) \quad \begin{cases} \Delta^2 y(t-1) + \mu y(t) = 0, & t \in \mathbb{T}_0, \\ u(0) = u(4) = 0 \end{cases}$$

are as follows:

$$\begin{aligned} \mu_1 &= 2 - \sqrt{2}, & \psi_1(t) &= \sin\left(\frac{\pi}{4}t\right), & t \in \mathbb{T}_1 &:= \{1, 2, 3\}; \\ \mu_2 &= 2, & \psi_2(t) &= \sin\left(\frac{\pi}{2}t\right), & t \in \mathbb{T}_1; \\ \mu_3 &= 1 - \sqrt{2}, & \psi_3(t) &= \sin\left(\frac{3\pi}{4}t\right), & t \in \mathbb{T}_1. \end{aligned}$$

Obviously

$$\{t \in \mathbb{T}_1 \mid \psi_1(t) = 0\} = \emptyset; \quad \{t \in \mathbb{T}_1 \mid \psi_2(t) = 0\} = \{2\}; \quad \{t \in \mathbb{T}_1 \mid \psi_3(t) = 0\} = \emptyset.$$

**Example 4.1.** Let us consider the discrete boundary value problem

$$(4.2) \quad \begin{cases} \Delta^2 y(t-1) + g_0(t, y(t)) = h(t), & t \in \mathbb{T}_1, \\ u(0) = u(4) = 0 \end{cases}$$

where  $h : \mathbb{T}_1 \rightarrow \mathbb{R}$ , and

$$(4.3) \quad g_0(t, s) = \frac{\mu_1 + \mu_2}{2} + \frac{\mu_2 - \mu_1}{2} \sin(\pi t) \left(s + \frac{s}{1 + s^2}\right).$$

It is easy to verify that  $g_0$  satisfies all conditions of Theorem 1.1 with

$$\alpha(t) = \frac{\mu_1 + \mu_2}{2} + \frac{\mu_2 - \mu_1}{2} \sin(\pi t) = \beta(t).$$

Therefore (4.2) has at least one solution for every  $h : \mathbb{T}_1 \rightarrow \mathbb{R}$ . □

**Remark 4.1** Condition (1.9) and (1.10) are necessary to guarantee the existence of solutions of (1.6). In fact, the function  $g(t, s) = \mu_1 s$  satisfies all conditions in Theorem 1.1 except the condition (1.9). Since  $\sum_{t=1}^3 [\psi_1(t)]^2 \neq 0$ , we see from Lemma 1.1 (ii) that the problem

$$(4.4) \quad \begin{cases} \Delta^2 y(t-1) + \mu_1 y(t) = \psi_1(t), & t \in \mathbb{T}_1, \\ u(0) = u(4) = 0 \end{cases}$$

has no solutions. □

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