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WEAK COMPACTNESS OF WEAK SOLUTIONS TO BACKWARD STOCHASTIC DIFFERENTIAL INCLUSIONS

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ABSTRACT. Weak compactness with respect to weak converegence in the Meyer-Zheng topology of sets of all weak solutions to backward stochastic differential inclusions is considered. Some existence theorems for backward stochastic differential inclusions are also given.

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1. INTRODUCTION

Given measurable and uniformly integrable bounded set-valued mappings $F : [0,T]\times\mathbb{R}^d\times\mathbb{R}^m \to Cl(\mathbb{R}^m)$ and $H : [0,T]\times\mathbb{R}^m \to Cl(\mathbb{R}^m)$ by a backward stochastic differential inclusion $BSDI(F,H)$ we mean relations

$$(1.1)\qquad \begin{cases} x_s \in E\left[x_t + \int_s^t F(\tau, x_\tau, z_\tau)d\tau|\mathcal{F}_s\right] \\ x_T \in \int_0^T H(t, z_t)dt \end{cases}$$

that have to be satisfied a.s. for every $0 \le s \le t \le T$ by a pair (x, z) of cádlág processes $x = (x_t)_{0\le t\le T}$ and $z = (z_t)_{0\le t\le T}$ defined on a complete filtered probability space $\mathcal{P}_{\mathbb{F}} = (\Omega, \mathcal{F}, P, \mathbb{F})$ with a filtration $\mathbb{F} = (\mathcal{F}_t)_{0\le t\le T}$ satisfying the usual hypothesis (see [11]). $E[x_t + \int_s^t F(\tau, x_\tau, z_\tau)d\tau|\mathcal{F}_s]$ denotes the set-valued conditional expectation (see [3], [4]) of the set-valued mapping $\Omega \ni \omega \longrightarrow x_t(\omega) + \int_s^t F(\tau, x_\tau(\omega), z_\tau(\omega))d\tau \subset \mathbb{R}^m$ with respect to the sub-σ-algebra $\mathcal{F}_s \subset \mathcal{F}$. If $\mathcal{P}_{\mathbb{F}}$ and a cádlág process z are given then x, satisfying conditions (1.1) with a filtration $\mathbb{F}^z$ generated by the process z is said to be a strong solution to $BSDI(F,H)$ with a driving process z. Usually the driving process z is an m-dimensional Browian motion or a strong solution of a forward stochastic differential equation. Let us recall that we call multifunctions F and H uniformly p-integrably bounded if there is a $m \in L^p([0,T], \mathbb{R}^+)$ such that $\max[h(F(t,x,z),\{0\}), h(H(t,z),\{0\})] \le m(t)$ for $(x,z) \in \mathbb{R}^d\times\mathbb{R}^m$ and a.e. $t \in [0,T]$, where h denotes the Hausdorff metric (see [7]). In a general case for given multifunctions F and H and a probability measure μ on a Borel σ-algebra of $D(\mathbb{R}^m)$ we can look for systems $(\mathcal{P}_{\mathbb{F}}, x, z)$ such that

(i) $Pz^{-1} = \mu$,

(ii) every $\mathbb{F}^z$-martingale is also $\mathbb{F}$-martingale,

(iii) a pair (x, z) satisfies (1) on $\mathcal{P}_{\mathbb{F}}$ a.s. for every $0 \leq s \leq t \leq T$.

Such systems are said to be weak solutions to $BSDI(F, H)$ with a given distribution μ of the driving process. In what follows we denote the $BSDI(F, H)$ with a given driving process z or with a given distribution μ of this process by $BSDI(F, H, z)$ and $BSDI(F, H, \mu)$, respectively. It is clear that if x is a strong solution to $BSDI(F, H, z)$ on $\mathcal{P}_{\mathbb{F}^z}$, then a system $(\mathcal{P}_{\mathbb{F}^z}, x, z)$ is a weak solution to $BSDI(F, H, Pz^{-1})$. The backward stochastic differential inclusions considered in this paper generalize the backward stochastic differential equations considered in [1] and the backward stochastic differential inclusions with continuous solutions considered in [8]. In some special cases the $BSDI(F, H)$ describes a class of recursive utilities under uncertainty (see [5] and [8]). The existence of weak solutions to $BSDI(F, H, \mu)$ and weak compactness of the set of all such its solutions need some special topology on the space $D(\mathbb{R}^{d+m})$ introduced by Meyer and Zheng in [10]. We present it in the Section 2. Some properties of Aumann's integrals and their conditional expections are given in the Section 3. Some existence theorems for $BSDI(F, H)$ are contained in the Section 4. The main result of the paper, dealling with the weak compactness of the set of all weak solutions to $BSDI(F, H)$ with respect to the Meyer-Zheng weak topology, is given in Section 5.

Throughout the paper we denote by $\mathcal{P}_{\mathbb{F}}$ a complete filtered probability space $(\Omega, \mathcal{F}\, P, \mathbb{F})$ with a filtration $\mathbb{F} = (\mathcal{F}_t)_{0\leq t\leq T}$ satisfying the usual hypotheses. Given $\mathcal{P}_{\mathbb{F}}$ we denote by $\mathbb{D}(\mathbb{F}, \mathbb{R}^m)$ the space of all m-dimensional $\mathbb{F}$-adapted cádlág processes on $\mathcal{P}_{\mathbb{F}}$ and by $\mathcal{S}^2(\mathbb{F}, \mathbb{R}^m)$ the set of all m-dimensional $\mathbb{F}$-semimartingales x such that $||x||^2_{\mathcal{S}^2} = E[\sup_{s\in[0,T]}|x_s|^2] < \infty$. We have $\mathcal{S}^2(\mathbb{F}, \mathbb{R}^m) \subset \mathbb{D}(\mathbb{F}, \mathbb{R}^m)$. It can be proved (see [11], Th.IV2.1.,Th.V.2.2.) that $(\mathcal{S}^2(\mathbb{F}, \mathbb{R}^m), || \cdot ||_{\mathcal{S}^2})$ is a Banach space.

2. THE MEYER-ZHENG TOPOLOGY

Let $T > 0$ be given and let $D(\mathbb{R}^k) = D([0, T], \mathbb{R}^k)$ denote the space of all cádlág functions $x : [0, T] \to \mathbb{R}^k$, i.e. every $x \in D(\mathbb{R}^k)$ is right continuous with left-hand limits such that $x(T) = \lim_{t\nearrow T} x(t)$, and by the convention $x(0-) = 0$. It is well known (see [7], Th.IV.1.14) that there is a metrizable topology on $D(\mathbb{R}^k)$, called the Skorokhod topology, for which this space is a Polish space. On the other hand there are not very much functions defined on $D(\mathbb{R}^k)$ that are continuous for the Skorokhod topology. For instance, the coordinate mapping π defined by $\pi_t(x) = x(t)$ for $x \in D(\mathbb{R}^k)$ is not continuous on $D(\mathbb{R}^k)$. Hence in particular, it follows that a function $g(t, \cdot)$ defined for fixed $t \in [0, T]$ by setting $g(t, x) = f(x(t))$ for $x \in D(\mathbb{R}^k)$ is not in general continuous on $D(\mathbb{R}^k)$ for a given continuous function $f : \mathbb{R}^k \to \mathbb{R}$. Therefore we are interested in introducing on $D(\mathbb{R}^k)$ another topology by which much more functions defined on $D(\mathbb{R}^k)$ is continuous. Such topology was defined on $D(\mathbb{R}^k)$

by Meyer and Zheng (see [10]) as the pseudopath topology. It was proved by Meyer and Zheng (see [10], Lemma 1) that the convergence of sequences in the pseudopath topology on $D(\mathbb{R}^k)$ is just convergence in the Lebesgue measure. Therefore (see [1], [11]) the pseudopath topology is metrizable and a compatible matric is given by

$$\rho(x,y) = \int_0^T (|x(t) - y(t)| \wedge 1)dt \tag{2.1}$$

for $x, y \in D(\mathbb{R}^k)$. The topology induced by this metric is called in [1] as the Meyer-Zheng topology on $D(\mathbb{R}^k)$. It can be verified (see [10], p. 355–356) that $(D(\mathbb{R}^k), \rho)$ is a separable metric space. But it is not complete. However (see [1], p. 33) it is a Lusin space. Then for every embedding in a compact metric space K, $D(\mathbb{R}^k)$ is a Borel set in K. It is easy to see that for every continuous function $f : \mathbb{R}^k \to L([0,T], \mathbb{R}^k)$ a function $f \circ \pi : D(\mathbb{R}^k) \to L([0,T], \mathbb{R}^k)$ is continuous for the Meyer-Zheng topology. In particular, a function

$$Y_t^k(x) = \frac{1}{\delta} \int_t^{T \wedge (t+\delta)} \pi_\tau^k(x) d\tau$$

with $\pi_t^k(x) = \pi_t(x)[(k + 1 - |\pi_t(x)|)^+ \wedge 1]$ for $k \geq 1$, $t \in [0,T]$ and $x \in D(\mathbb{R}^d)$ is bounded and continuous in the Meyer-Zheng topology for every $\delta > 0$.

Similarly as in [1] we will consider $D(\mathbb{R}^k)$ as a measurable space with a natural σ-algebra $\mathcal{D}(\mathbb{R}^k)$ defined by the projection π, i.e. with $\mathcal{D}(\mathbb{R}^k) = \sigma(\{\pi_u : 0 \leq u \leq T\})$. Similarly we define σ - algebras $\mathcal{D}_t(\mathbb{R}^k) = \sigma(\{\pi_u : 0 \leq u \leq t\})$ and $\mathcal{D}_t^T(\mathbb{R}^k) = \sigma(\{\pi_u : t \leq u \leq T\})$ for fixed $0 \leq t \leq T$. Throughout the paper we shall assume that the natural filtrations $(\mathcal{D}_t(\mathbb{R}^k))_{0 \leq t \leq T}$ and $(\mathcal{D}_t^T(\mathbb{R}^k))_{t \leq t \leq T}$ are augmented and satisfy the usual hypotheses. We have the following result.

Proposition 2.1 ([1], Lemma 4.2.). *Let $\beta(D(\mathbb{R}^k))$ be the Borel σ-algebra of Borel subsets of $D(\mathbb{R}^k)$ in the Meyer-Zheng topology. Then $\beta(D(\mathbb{R}^k)) = \mathcal{D}(\mathbb{R}^k)$.*

In what follows we shall consider $D(\mathbb{R}^k)$ with $k = d + m$, i.e. $D(\mathbb{R}^{d+m}) = D(\mathbb{R}^d) \times D(\mathbb{R}^m)$. In such a case by π^d and π^m we denote partial coordinate mappings defined on $D(\mathbb{R}^{d+m})$ by settings $\pi_t^d(x,y) = x(t)$ and $\pi_t^m(x,y) = y(t)$ for fixed $0 \leq t \leq T$ and $(x,y) \in D(\mathbb{R}^{d+m})$.

3. SOME PROPERTIES OF AUMANN'S INTEGRALS AND ITS CONDITIONAL EXPECTION

Given a measurable set-valued mapping $G : [0,T] \to Cl(\mathbb{R}^m)$, where $Cl(\mathbb{R}^m)$ denotes a family of all nonempty closed subsets of $\mathbb{R}^m$, we denote by $S(G)$ the set of all Lebesgue integrable selectors for G, i.e. Lebesgue integrable functions $g : [0,T] \to \mathbb{R}^m$ such that $g(t) \in G(t)$ for a.e. $t \in [0,T]$. If $S(G) \neq \emptyset$ then G is said to be Aumann integrable and a family $\{\int_0^T g(t)dt : g \in S(G)\}$ is denoted by $\int_0^T G(t)dt$

and called the Aumann's integral of G on $[0,T]$. Immediately from Kuratowski and Ryll-Nardzewski measurable selection theorem (see [9], [6]) it follows that if G is measurable and integrable bounded then it is Aumann integrable. We shall need the following properties of Aumann's integrals.

Proposition 3.1 ([6], Th. II.3.20). *Let $G : [0,T] \to Cl(\mathbb{R}^m)$ be Aumann integrable. Then $\int_0^T G(t)dt = \int_0^T coG(t)dt$ and both integrals are convex and compact subsets of $\mathbb{R}^m$.*

Proposition 3.2 ([6], Th. II.3.21). *Let $G : [0,T] \to Cl(\mathbb{R}^m)$ be Aumann integrable. Then $\int_{\mathcal{U}} \sigma(p, G(t))dt = \sigma(p, \int_{\mathcal{U}} G(t)dt)$ for every $p \in \mathbb{R}^m$ and a measurable set $\mathcal{U} \subset [0,T]$, where $\sigma(p,\cdot)$ denotes the support function on $\mathbb{R}^m$, where $coG(t)$ denotes the convex hull of $G(t)$.*

It can be verified ([6], Th. III.1.2) that a space $\mathcal{A}$ (equivalence classes of) all Aumann integrable set-valued mappings is a complete metric space with a metric d defined by $d(F,G) = \int_0^T h(F(t), G(t))dt$ for $F, G \in \mathcal{A}$, where h is the Hausdorff metric on the space $Comp(\mathbb{R}^m)$ of all nonempty compact subsets of $\mathbb{R}^m$. In what follows we shall deal with set-valued mappings $F : [0,T] \times X \to Cl(\mathbb{R}^m)$ where (X, ρ) is a given metric space. If $F(\cdot, x)$ is measurable and uniformly integrable bounded then we can define a set-valued mapping $S(F) : X \to \mathcal{P}(L([0,T], \mathbb{R}^m))$ by setting $S(F)(x) = \{f \in L([0,T], \mathbb{R}^m) : f(t) \in F(t,x) \ a.e.\}$ for $x \in X$, where $\mathcal{P}(L([0,T], \mathbb{R}^m))$ denotes a space of all nonempty subsets of $L([0,T], \mathbb{R}^m)$. We say that F is $\mathcal{A}$-continuous with respect to its second argument if a mapping $X \ni x \to F(\cdot, x) \in \mathcal{A}$ is continuous as a mapping from (X, ρ) into $(\mathcal{A}, d)$. It can be verified (see [6], Lemma III.2.8.) that $S(F)$ is continuous if F is $\mathcal{A}$-continuous with respect to its second variable. We say that F is $\mathcal{A}$-lower semicontinuous with respect to its second variable if for every $x \in X$ and every sequence $(x_n)_{n=1}^{\infty}$ of (X, ρ) converging to x one has $\lim_{n\to\infty} \int_0^T \sup_{u \in F(t,x)} dist(u, F(t, x_n))dt = 0$. It can be proved (see [6], Lemma III.2.9.) that if F is $\mathcal{A}$-lower semicontinuous with respect to its second variable then $S(F)$ is lower semicontinuous on X. Furthermore (see [2], Th. 42) if (X, ρ) is separable then $S(F)$ admits an $\mathcal{A}$-continuous selector.

Let $(\Omega, \mathcal{F}, P)$ be a probability space and let $\mathcal{G}$ be a sub-σ-algebra of $\mathcal{F}$. Given an $\mathcal{F}$-measurable set-valued mapping $\Phi : \Omega \to Cl(\mathbb{R}^m)$ with a nonempty set $S(\Phi)$ of all its $\mathcal{F}$-measurable and integrable selectors there exists (see [3], [4]) an unique (in the a.s. sense) $\mathcal{G}$- measurable set-valued mapping $E[\Phi|\mathcal{G}]$ satisfying

$$S(E[\Phi|\mathcal{G}]) = cl_L\{E[\varphi|\mathcal{G}] : \varphi \in S(\Phi)\} \tag{3.1}$$

where cl_L denotes the closure operation in $L(\Omega, \mathcal{G}, \mathbb{R}^m)$. We call $E[\Phi|\mathcal{G}]$ the multi-valued conditional expectation of Φ relative to $\mathcal{G}$. This conditional expectation has properties similar to those of the usual ones. For example, we have $\int_A E[\Phi|\mathcal{G}]dP =$

$\int_A \Phi dP$ for every $A \in \mathcal{G}$, where integrals are understood in the Aumann's sense (see [4], Prop. 6.8). It can be proved (see [4], Prop. 6.2.) that for given measurable and integrably bounded set-valued mappings $\Phi, \Psi : \Omega \to Cl(\mathbb{R}^m)$ one has $Eh(E[\Phi|\mathcal{G}], E[\Psi|\mathcal{G}]) \leq Eh(\Phi, \Psi)$.

Let $G : [0,T] \times \Omega \to Cl(\mathbb{R}^m)$ be measurable and integrably bounded. Similarly as above we denote by $S(G)$ the set of all integrable selectors of G. It is easy to verify (see [6]) that $S(G)$ is nonempty and decomposable, i.e. that for every $f, g \in S(G)$ and $E \in \beta_T \otimes \mathcal{F}$ one has $\mathbb{1}_E f + \mathbb{1}_{E^\sim} g \in S(G)$, where β_T denotes the Borel σ-algebra of $[0,T]$ and $E^\sim$ is the complement of E. In particular, if $G(t,\omega)$ are convex subsets of $\mathbb{R}^m$ for $(t,\omega) \in [0,T] \times \Omega$, the set $S(G)$ is a convex weakly compact subset of $L([0,T] \times \Omega, \mathbb{R}^m)$. Then it is also a closed subset of this space. For the given above G we can define an Aumann integral $\Phi(\omega) = \int_0^T G(t,\omega)dt$ depending on a parameter $\omega \in \Omega$. By virtue of Lemma 2 a set-valued integral $\int_0^T G(t,\omega)dt$ is a nonempty convex compact subset of $\mathbb{R}^m$ for every $\omega \in \Omega$. Furthermore, $\int_0^T G(t,\omega)dt = \int_0^T co\, G(t,\omega)dt$ for $\omega \in \Omega$. Hence and Lemma 3 we obtain the following result.

Proposition 3.3. *Let $G : [0,T] \times \Omega \to Cl(\mathbb{R}^m)$ be measurable and integrably bounded. Then a set-valued mapping $\Phi : \Omega \to \mathrm{Conv}(\mathbb{R}^m)$ defined by $\Phi(\omega) = \int_0^T G(t,\omega)dt$ for $\omega \in \Omega$ is measurable.*

Proof. By virtue of ([6],Th. II.3.8) it is enough only to verify that the function $\Omega \ni \omega \to s(p, \Phi(\omega)) \in \mathbb{R}$ is measurable for every $p \in \mathbb{R}^n$, where $s(\cdot, A)$ denotes a support function of $A \in Cl(\mathbb{R}^m)$. By the measurability of G and its integrably boundedness a function $[0,T] \times \Omega \ni (t,\omega) \to s(p, G(t,\omega)) \subset \mathbb{R}$ is measurable for every $p \in \mathbb{R}^m$ (see [8], Remark II.3.5). By virtue of Proposition 3.2 for every $p \in \mathbb{R}^m$ one has $s(p, \Phi(\omega)) = \int_0^T s(p, G(t,\omega))dt$ for $\omega \in \Omega$. Hence the measurability of the function $\Omega \ni \omega \to s(p, \Phi(\omega)) \in \mathbb{R}$ follows for every $p \in \mathbb{R}^m$. Therefore Φ is $\mathcal{F}$-measurable. $\square$

Proposition 3.4. *Let $G : [0,T] \times \Omega \to Cl(\mathbb{R}^m)$ be measurable and integrably bounded and let $\Phi(\omega) = \int_0^T G(t,\omega)dt$ for $\omega \in \Omega$. Then $S(\Phi)$ is a nonempty convex weakly compact subset of $L(\Omega, \mathcal{F}, \mathbb{R}^m)$. Furthermore, $\varphi \in S(\Phi)$ if and only if there is $g \in S(co\, G)$ such that $\varphi(\omega) = \int_0^T g(t,\omega)dt$ for a.e. $\omega \in \Omega$.*

Proof. By Proposition 3.3, Φ is $\mathcal{F}$-measurable. It is also integrably bounded, because $\|\Phi(\omega)\| \leq \int_0^T m(t,\omega)dt$ for a.e. $\omega \in \Omega$. Therefore (see [6], Th. III.2.3) $S(\Phi)$ is a nonempty convex weakly compact subset of $L(\Omega, \mathcal{F}, \mathbb{R}^m)$. For every $g \in S(co\ G)$ a function $\varphi(\omega) = \int_0^T g(t,\omega)dt$ is a measurable selector for Φ, because of Proposition 3.1 we have $\Phi(\omega) = \int_0^T co\ G(t,\omega)dt$ for $\omega \in \Omega$. It is also integrably bounded, because $|\varphi(\omega)| \leq \int_0^T m(t,\omega)dt$ for a.e. $\omega \in \Omega$. Then $\varphi \in S(\Phi)$ for every $g \in S(co\, G)$. Assume now $\varphi \in S(\Phi)$. Then for every $A \in \mathcal{F}$ one has $E_A\varphi \in E_A\Phi$, where $E_A\varphi = \int_A \varphi dP$ and $E_A\Phi = \int_A \Phi dP$. Let $\varepsilon > 0$ be given and select a measurable partition $(A_n^\varepsilon)_{n=1}^{N_\varepsilon}$ of

Ω such that $E_{A_n^\varepsilon}\int_0^T m(t,\cdot)dt < \varepsilon/2^{n+1}$. For every $n = 1,\dots,N_\varepsilon$ there is a $g_n^\varepsilon \in S(G)$ such that $E_{A_n^\varepsilon}\varphi = E_{A_n^\varepsilon}\int_0^T g_n^\varepsilon(t,\cdot)dt$. Let $g^\varepsilon = \sum_{n=1}^{N_\varepsilon} \mathbb{1}_{A_n^\varepsilon} g_n^\varepsilon$. By the decomposability of $S(G)$ one has $g^\varepsilon \in S(G)$. We have $g^\varepsilon \in S(co\, G)$ because $S(G) \subset S(co\, G)$. Taking a sequence $(\varepsilon_k)_{k=1}^\infty$ of positive numbers converging to zero we can select $g \in S(co\, G)$ and a subsequence, denoted again by $(g^{\varepsilon_k})_{k=1}^\infty$, of $(g^{\varepsilon_k})_{k=1}^\infty$ weakly converging to g in $L([0,T]\times\Omega,\mathbb{R}^n)$, because $S(co\, G)$ is a weakly compact subset of $L([0,T]\times\Omega,\mathbb{R}^n)$. For every $A \in \mathcal{F}$ and $k = 1,2,\dots$ there is a subset $\{n_1,\dots,n_p\}$ of $\{1,\dots,N_{\varepsilon_k}\}$ such that $A\cap A_{n_i}^{\varepsilon_k} \neq \emptyset$ for $i = 1,2,\dots,p$ and $A \cap A_r = \emptyset$ for $r \in \{1,2,\dots,N_{\varepsilon_k}\}\setminus\{n_1,\dots,n_p\}$. Therefore

$$\begin{aligned}
&\left|E_A\varphi - E_A\int_0^T g^{\varepsilon_k}(t,\cdot)dt\right| \leq \\
&\leq \sum_{n=1}^{N_{\varepsilon_k}} \left|E_{A\cap A_n^{\varepsilon_k}}\varphi - E_{A\cap A_n^{\varepsilon_k}}\int_0^T g_n^{\varepsilon_k}(t,\cdot)dt\right| = \\
&= \sum_{i=1}^{p} \left|E_{A\cap A_{n_i}^{\varepsilon_k}}\varphi - E_{A\cap A_{n_i}^{\varepsilon_k}}\int_0^T g_n^{\varepsilon_k}(t,\cdot)dt\right| \leq \\
&\leq 2\sum_{i=1}^{p} E_{A_{n_i}^{\varepsilon_k}}\int_0^T m(t,\cdot)dt \leq \varepsilon_k
\end{aligned}$$

for every $k = 1,2,\dots$. On the other hand for every $A \in \mathcal{F}$ we also have

$$\begin{aligned}
&\left|E_A\varphi - E_A\int_0^T g(t,\cdot)dt\right| \leq \\
&\leq \left|E_A\varphi - E_A\int_0^T g^{\varepsilon_k}(t,\cdot)dt\right| + \left|E_A\int_0^T g^{\varepsilon_k}(t,\cdot)dt - E_A\int_0^T g(t,\cdot)dt\right| \\
&\leq \varepsilon_k + \left|E_A\int_0^T g^{\varepsilon_k}(t,\cdot)dt - E_A\int_0^T g(t,\cdot)dt\right|
\end{aligned}$$

for $k = 1,2,\dots$. Hence it follows that $E_A\varphi = E_A\int_0^T g(t,\cdot)dt$ for every $A \in \mathcal{F}$, because $\varepsilon_k \to 0$ and $|E_A\int_0^T g^{\varepsilon_k}(t,\cdot)dt - E_A\int_0^T g(t,\cdot)dt| \to 0$ as $k \to \infty$. Therefore $\varphi(\omega) = \int_0^T g(t,\cdot)dt$ for a.e. $\omega \in \Omega$. $\square$

Corollary 3.5. *If $G : [0,T]\times\Omega \to Cl(\mathbb{R}^m)$ is measurable and integrably bounded then*

$$S\left(\int_0^T G(t,\cdot)dt\right) = \left\{\int_0^T g(t,\cdot)dt : g \in S(co\, G)\right\}.$$

Corollary 3.6. *If $G : [0,T]\times\Omega \to Cl(\mathbb{R}^m)$ is measurable and integrably bounded and $\mathcal{G}$ is a sub-σ-algebra of $\mathcal{F}$ then*

$$S\left(E\left[\int_0^T G(t,\cdot)dt|\mathcal{G}\right]\right) = \left\{E\left[\int_0^T g(t,\cdot)dt|\mathcal{G}\right] : g \in S(co\, G)\right.$$

Proof. It is enough only to see that the set $\mathcal{H} = \{E[\int_0^T g(t,\cdot)dt|\mathcal{G}] : g \in S(co\, G)\}$ is a closed subset of $L(\Omega, \mathcal{G}, \mathbb{R}^m)$. By the properties of conditional expectations and the properties of the set $S(co\, G)$ it follows that $\mathcal{H}$ is a convex weakly compact subset of $L(\Omega, \mathcal{G}, \mathbb{R}^m)$. Therefore $\mathcal{H}$ is a closed subset of $L(\Omega, \mathcal{G}, \mathbb{R}^m)$. □

4. MEASURABLE SELECTION THEOREM

Let $x = (x_t)_{0 \le t \le T}$ be an $\mathbb{F}$-adapted m-dimensional cádlág process on $\mathcal{P}_{\mathbb{F}}$. Given a measurable, $\mathbb{F}$-adapted and integrably bounded multivalued mapping $G : [0,T] \times \Omega \to Cl(\mathbb{R}^m)$ we denote by $S_{\mathbb{F}}(G)$ a set of all measurable and $\mathbb{F}$-adapted selectors for G. Let us observe that G is measurable and $\mathbb{F}$-adapted if and only if it is $\Sigma_{\mathbb{F}}$-measurable, where $\Sigma_{\mathbb{F}} = \{A \in \beta_T \otimes \mathcal{F} : A_t \in \mathcal{F}_t \text{ for } 0 \le t \le T\}$ and A_t denotes a section of a set $A \in \beta_T \otimes \mathcal{F}$ at $t \in [0,T]$. Therefore, immediately from Kuratowski and Ryll-Nardzewski measurable selection theorem (see [6], Th. II.3.10) it follows that for the given above G the set $S_{\mathbb{F}}(G)$ is nonempty. Similarly as above we can verify that $S_{\mathbb{F}}(co\, G)$ is a nonempty convex and weakly compact subset of $L([0,T] \times \Omega, \Sigma_{\mathbb{F}}, \mathbb{R}^m)$. We have proved in [8] the following measurable selection theorem.

Theorem 4.1. *Let $G : [0,T] \times \Omega \to Cl(\mathbb{R}^m)$ be a measurable $\mathbb{F}$-adapted and integrably bounded set-valued mapping. Assume $x = (x_t)_{0 \le t \le T}$ is an m-dimensional measurable process on $\mathcal{P}_{\mathbb{F}}$ such that $E|x_T| < \infty$. Then*

$$x_s \in E\left[x_t + \int_s^t G(\tau,\cdot)d\tau|\mathcal{F}_s\right] \qquad a.s. \tag{4.1}$$

for every $0 \le s \le t \le T$ if and only if there is $g \in S_{\mathbb{F}}(co\, G)$ such that

$$x_t = E\left[x_T + \int_t^T g(\tau,\cdot)d\tau|\mathcal{F}_t\right] \qquad a.s. \tag{4.2}$$

for every $0 \le t \le T$.

Proof. Suppose there is $g \in S_{\mathbb{F}}(co\ G)$ such that (4.2) is satisfied. Then for every $0 \le s \le t \le T$ one has

$$x_s = E\left[x_T + \int_s^T g(\tau,\cdot)d\tau|\mathcal{F}_s\right] = E\left[\int_s^t g(\tau,\cdot)d\tau|\mathcal{F}_s\right]$$
$$+E\left[x_T + \int_t^T g(\tau,\cdot)d\tau|\mathcal{F}_s\right]$$

and

$$E[x_t|\mathcal{F}_s] = E\left[x_T + \int_t^T g(\tau,\cdot)d\tau|\mathcal{F}_s\right]$$

a.s. Therefore

$$x_s = E\left[x_t + \int_s^t g(\tau,\cdot)d\tau|\mathcal{F}_s\right]$$

a.s. for $0 \le s \le t \le T$. Hence by Corollary 3.6 it follows that

$$x_s \in S\left(E\left[x_t + \int_s^t G(\tau,\cdot)d\tau|\mathcal{F}_s\right]\right)$$

for $0 \le s \le t \le T$. Therefore, (4.1) is satisfied a.s. for $0 \le s \le t \le T$. □

Assume that (4.1) is satisfied for every $0 \le s \le t \le T$ a.s. and let $m \in L([0,T]\times\Omega,\mathbb{R}_+)$ be such that $\|G(t,\omega)\| \le m(t,\omega)$ for a.e. $(t,\omega) \in [0,T]\times\Omega$. For every $0 \le t \le T$ one has $E|x_t| \le E|x_T| + E\int_0^T m(t,\cdot)dt < \infty$. By virtue of Corollary 3.6, a process x is $\mathbb{F}$-adapted. Let $\eta > 0$ be arbitrarily fixed and select $\delta > 0$ such that $\delta < T$ and $\sup_{0\le t\le T-\delta} E\int_t^{t+\delta} m(\tau,\cdot)d\tau < \eta/2$. For fixed $t \in [0,T-\delta]$ and $t \le \tau \le t+\delta$ we have $x_t \in E[x_\tau + \int_t^\tau G(s,\cdot)ds|\mathcal{F}_t]$ a.s. Therefore, for every $A \in \mathcal{F}_t$ we get $E_A(x_t - x_\tau) \in E_A\int_t^\tau G(s,\cdot)ds$. Then $|E_A(x_t - x_\tau)| \le E_A\int_t^\tau \|G(s,\cdot)\|ds \le E\int_t^{t+\delta} m(s,\cdot)ds < \eta/2$ for every $0 \le t \le T-\delta$ and $A \in \mathcal{F}_t$. Therefore, $\sup_{t\le\tau\le t+\delta}|E_A(x_t - x_\tau)| \le \eta/2$ for every $A \in \mathcal{F}_t$ and fixed $0 \le t \le T-\delta$.

Let $\tau_0 = 0$, $\tau_1 = \delta, \ldots, \tau_{N-1} = (N-1)\delta < T \le N\delta$. Immediately from (4.1) and Corollary 3.6 it follows that for every $i = 1,2,\ldots,N-1$ there is $g_i^\eta \in S_{\mathbb{F}}(co\, G)$ such that

$$E\left|x_{\tau_{i-1}} - E\left[x_{\tau_i} + \int_{\tau_{i-1}}^{\tau_i} g_i^\eta(s,\cdot)ds|\mathcal{F}_{\tau_{i-1}}\right]\right| = 0.$$

Furthermore, there is $g_N^\eta \in S_{\mathbb{F}}(co\, G)$ such that

$$E\left|x_{\tau_{N-1}} - E\left[x_T + \int_{\tau_{N-1}}^{T} g_N^\eta(s,\cdot)ds|\mathcal{F}_{\tau_{N-1}}\right]\right| = 0.$$

Define $g^\eta = \sum_{i=1}^{N-1}\mathbb{1}_{[\tau_{i-1},\tau_i)}g_i^\eta + \mathbb{1}_{[\tau_{N-1},T]}g_N^\eta$. By the decomposability of $S_{\mathbb{F}}(co\, G)$ we have $g^\eta \in S_{\mathbb{F}}(co\, G)$. For fixed $t \in [0,T]$ there is $p \in \{1,2,\ldots,N-1\}$ or $p = N$ such that $t \in [\tau_{p-1},\tau_p)$ or $t \in [\tau_{N-1},T]$. Let $t \in [\tau_{p-1},\tau_p)$ with $1 \le p \le N-1$. For every $A \in \mathcal{F}_t$ one has

$$\begin{aligned}
&\left|E_A\left(x_t - E\left[x_T + \int_t^T g^\eta(s,\cdot)ds\mathcal{F}_t\right]\right)\right| \\
&\le |E_A(x_t - x_{\tau_p})| + E\left|x_{\tau_p} - E\left[x_{\tau_{p+1}} + \int_{\tau_p}^{\tau_{p+1}} g^\eta(s,\cdot)d\tau|\mathcal{F}_{\tau_p}\right]\right| \\
&\quad + |E_A(E[x_{\tau_{p+1}}|\mathcal{F}_{\tau_p}] - x_{\tau_{p+1}})| + E\left|\int_t^{\tau_p} g^\eta(s,\cdot)ds\right| + \\
&\quad + \left|E_A\left(E\left[\int_{\tau_p}^{\tau_{p+1}} g^\eta(s,\cdot)ds|\mathcal{F}_{\tau_p}\right] - E\left[\int_{\tau_p}^{\tau_{p+1}} g^\eta(s,\cdot)d\tau|\mathcal{F}_t\right]\right)\right| + \cdots + \\
&\quad + E\left|x_{\tau_{N-1}} - E\left[x_T + \int_{\tau_{N-1}}^{T} g^\eta(s,\cdot)d\tau|\mathcal{F}_{\tau_{N-1}}\right]\right| \\
&\quad + |E_A(E[x_{\tau_{N-1}}|\mathcal{F}_{\tau_{N-1}}] - x_{\tau_{N-1}})| + E_A\left(E\left[\int_{\tau_{N-1}}^{T} g^\eta(s,\cdot)ds|\mathcal{F}_{\tau_{N-1}}\right] -
\end{aligned}$$

$$
\begin{aligned}
&- E\Big[\int_{\tau_{N-1}}^{T} g^{\eta}(s,\cdot)ds|\mathcal{F}_t\Big]\Big)\Big| \leq \sup_{t\leq\tau\leq t+\delta}|E_A(x_t - x_\tau)| + E\int_t^{t+\delta} m(s,\cdot)ds + \\
&+ \sum_{i=p}^{N-2} E\Big|x_{\tau_i} - E\Big[x_{\tau_{i+1}} + \int_{\tau_i}^{\tau_{i+1}} g^{\eta}(s,\cdot)ds|\mathcal{F}_{\tau_i}\Big]\Big| \\
&+ E\Big|x_{\tau_{N-1}} - E\Big[x_T + \int_{\tau_{N-1}}^{T} g^{\eta}(s,\cdot)d\tau|\mathcal{F}_{\tau_{N-1}}\Big]\Big| \\
&+ \sum_{i=p}^{N-2}|E_A(E[x_{\tau_{i+1}}|\mathcal{F}_{\tau_i}] - x_{\tau_{i+1}})| + \sum_{i=p}^{N-2}\Big|E_A\Big(E\Big[\int_{\tau_i}^{\tau_{i+1}} g^{\eta}(s,\cdot)ds|\mathcal{F}_{\tau_i}\Big] - \\
&- E\Big[\int_{\tau_i}^{\tau_{i+1}} g^{\eta}(s,\cdot)ds|\mathcal{F}_t\Big]\Big)\Big| \\
&+ \Big|E_A\Big(E\Big[\int_{\tau_{N-1}}^{T} g^{\eta}(s,\cdot)ds|\mathcal{F}_{\tau_{N-1}}\Big] - E\Big[\int_{\tau_{N-1}}^{T} g^{\eta}(s,\cdot)ds|\mathcal{F}_t\Big]\Big)\Big|.
\end{aligned}
$$

But $\mathcal{F}_t \subset \mathcal{F}_{\tau_i}$ for $i = p, p+1, \ldots, N-1$. Then for $A \in \mathcal{F}_t$ one has

$$\sum_{i=p}^{N-2}|E_A(E[x_{\tau_{i+1}}|\mathcal{F}_{\tau_i}] - x_{\tau_{i+1}})| = 0,$$

$$\sum_{i=p}^{N-2}\Big|E_A\Big(E\left[\int_{\tau_i}^{\tau_{i+1}} g^{\eta}(s,\cdot)ds|\mathcal{F}_{\tau_i}\right] - E\left[\int_{\tau_i}^{\tau_{i+1}} g^{\eta}(s,\cdot)ds|\mathcal{F}_t\right]\Big)\Big| = 0$$

and

$$\Big|E_A\Big(E\Big[\int_{\tau_{N-1}}^{T} g^{\eta}(s,\cdot)ds|\mathcal{F}_{\tau_{N-1}}\Big] - E\Big[\int_{\tau_{N-1}}^{T} g^{\eta}(s,\cdot)ds|\mathcal{F}_t\Big]\Big)\Big| = 0.$$

Hence it follows

$$\left|E_A\left(x_t - E\left[x_T + \int_t^T g^{\eta}(s,\cdot)ds|\mathcal{F}_t\right]\right)\right| \leq \eta \tag{4.3}$$

for fixed $0 \leq t \leq T$ and $A \in \mathcal{F}_t$. Let $(\eta_j)_{j=1}^{\infty}$ be a sequence of positive numbers converging to zero. For every $j = 1, 2, \ldots$ we can select $g^{\eta_j} \in S_{\mathbb{F}}(co\ G)$ such that (4.3) is satisfied with $\eta = \eta_j$. By the weak compactness of $S_{\mathbb{F}}(co\ G)$ there are $g \in S_{\mathbb{F}}(co\ G)$ and a subsequence $(g^{\eta_k})_{k=1}^{\infty}$ of $g^{\eta_j})_{j=1}^{\infty}$ weakly converging to g in $L([0,T]\times\Omega, \Sigma_{\mathbb{F}}, \mathbb{R})$. Then for every $A \in \mathcal{F}_t \subset \mathcal{F}$ one has $\lim_{k\to\infty} E_A\int_t^T g^{\eta_k}(s,\cdot)ds = E_A\int_t^T g(s,\cdot)ds$. On the other hand for every fixed $t \in [0,T]$ and $A \in \mathcal{F}_t$ we have

$$
\begin{aligned}
&\Big|E_A\Big(x_t - E\Big[x_T + \int_t^T g(s,\cdot)ds|\mathcal{F}_t\Big]\Big)\Big| \\
&\leq \Big|E_A\Big(x_t - E\Big[x_T + \int_t^T g^{\eta_k}(s,\cdot)ds|\mathcal{F}_t\Big]\Big)\Big| \\
&\qquad + \Big|E_A\Big(E\Big[\int_t^T g^{\eta_k}(s,\cdot)ds|\mathcal{F}_t\Big] - E\Big[+\int_t^T g(s,\cdot)ds|\mathcal{F}_t\Big]\Big)\Big| \\
&\leq \eta_k + \Big|E_A\int_t^T g^{\eta_k}(s,\cdot)ds - E_A\int_t^T g(s,\cdot)ds\Big|
\end{aligned}
$$

for $k = 1, 2, \ldots$. Therefore

$$E_A\bigg(x_t - E\bigg[x_T + \int_t^T g(s,\cdot)ds|\mathcal{F}_t\bigg]\bigg) = 0$$

for every $A \in \mathcal{F}_t$ and fixed $0 \le t \le T$. But x_t and $E[x_T + \int_t^T g(s,\cdot)ds|\mathcal{F}_t]$ are $\mathcal{F}_t$-measurable. Then $x_t = E[x_T + \int_t^T g(s,\cdot)ds|\mathcal{F}_t]$, P-a.s., for $0 \le t \le T$.

5. **EXISTENCE THEOREMS**

Given measurable and uniformly integrable bounded set-valued mappings $F : [0,T] \times \mathbb{R}^d \times \mathbb{R}^m \to Cl(\mathbb{R}^m)$ and $H : [0,T] \times \mathbb{R}^m \to Cl(\mathbb{R}^d)$ and a pair $(x,z) \in \mathbb{D}(\mathbb{F}, \mathbb{R}^d) \times \mathbb{D}(\mathbb{F}, \mathbb{R}^m)$ with $\mathbb{F} = (\mathcal{F}_t)_{0\le t\le T}$ we denote by $S_{\mathbb{F}}(F)(x,z)$ the set of all measurable and $\mathbb{F}$-adapted selectors to the set-valued mapping $G_{xz} : [0,T] \times \Omega \to Cl(\mathbb{R}^d)$ defined by $G_{xz}(t,\omega) = F(t, x_t(\omega), z_t(\omega))$ for $0 \le t \le T$ and $\omega \in \Omega$. It is clear that G_{xz} is measurable and $\mathbb{F}$-adapted that is equivalent to $\Sigma_{\mathbb{F}}$-measurability. It is also integrable bounded. Then by Kuratowski and Ryll-Nardzewski measurable selection theorem we have $S_{\mathbb{F}}(F)(x,z) \ne \emptyset$ and $S_{\mathbb{F}}(F)(x,z) \subset L([0,T] \times \Omega, \Sigma_{\mathbb{F}}, \mathbb{R}^d)$. If F takes on convex values then $S_{\mathbb{F}}(F)(x,z)$ is a convex weakly compact subset of this space.

Proposition 5.1. *Let $F : [0,T] \times \mathbb{R}^d \times \mathbb{R}^m \to Cl(\mathbb{R}^m)$ be measurable and uniformly integrable bounded and let $(x,z) \in \mathbb{D}(\mathbb{F}, \mathbb{R}^d) \times \mathbb{D}(\mathbb{F}, \mathbb{R}^m)$. Assume $Y = (Y_t)_{0\le t\le T}$ is an d-dimensional measurable stochastic process on $\mathcal{P}_{\mathbb{F}}$ such that that $E|Y_T| < \infty$ and*

$$Y_s \in E[Y_t + \int_s^t F(\tau, x_\tau, z_\tau)d\tau|\mathcal{F}_s] \tag{5.1}$$

a.s for $0 \le s \le t \le T$. Then Y possesses an $\mathbb{F}$-cádlág version, denoted again by Y. Moreover Y is an $\mathbb{F}$-semimartingale and has the semimartingale decomposition $Y_t = Y_0 + M_t + A_t$ for $0 \le t \le T$, where $Y_0 = E[Y_T + \int_0^T f_\tau^{xz} d\tau|\mathcal{F}_0]$, $M_t = E[Y_T + \int_0^T f_\tau^{xz} d\tau|\mathcal{F}_t] - E[Y_T + \int_0^T f_\tau^{xz} d\tau|\mathcal{F}_0]$ and $A_t = -\int_0^t f_\tau^{xz} d\tau$ for $0 \le t \le T$ with $f^{xz} \in S_{\mathbb{F}}(co\ F)(x,z)$ such that $Y_t = E[Y_T + \int_t^T f_\tau^{xz} d\tau|\mathcal{F}_t]$ a.s. for $0 \le t \le T$.

Proof. By virtue of Theorem 4.1 there is $f^{xz} \in S_{\mathbb{F}}(co\ F)(x,z)$ such that $Y_t = E[Y_T + \int_t^T f_\tau^{xz} d\tau|\mathcal{F}_t]$ a.s. for $0 \le t \le T$. Hence, similarly as in [1] the result follows. □

Corollary 5.2. *Let $F : [0,T] \times \mathbb{R}^d \times \mathbb{R}^m \to Cl(\mathbb{R}^m)$ and $H : [0,T] \times \mathbb{R}^m \to Cl(\mathbb{R}^m)$ be measurable and uniformly integrable bounded and let μ be a probability measure on $\mathcal{D}(\mathbb{R}^m)$. If $(\mathcal{P}_{\mathbb{F}}, x, z)$ satisfies conditions (i) and (ii) of the definition of a weak solution to $BSDI(F,H,\mu)$ then $(\mathcal{P}_{\mathbb{F}}, x, z)$ is a weak solution to $BSDI(F,H,\mu)$ if and only if there are $f^{xz} \in S_{\mathbb{F}}(co\ F)(x,z)$ and $\xi^z \in S_{\mathbb{F}}(H)(z)$ such that $x_t = x_0 + M_t + A_t$ where $x_0 = E[\xi^z + \int_0^T f_\tau^{xz} d\tau|\mathcal{F}_0]$, $M_t = E[\xi^z + \int_0^T f_\tau^{xz} d\tau|\mathcal{F}_t] - E[\xi^z + \int_0^T f_\tau^{xz} d\tau|\mathcal{F}_0]$ and $A_t = -\int_0^t f_\tau^{xz} d\tau$ for $0 \le t \le T$.*

Similarly as in [8] we can prove the following existence theorem.

Theorem 5.3. *Let $\mathcal{P}_{\mathbb{F}}$ and $z \in \mathbb{D}(\mathbb{F}, \mathbb{R}^m)$ be given and let $F : [0,T] \times \mathbb{R}^d \times \mathbb{R}^m \to Cl(\mathbb{R}^m)$ and $H : [0,T] \times \mathbb{R}^m \to Cl(\mathbb{R}^m)$ be measurable and uniformly square integrable bounded. Assume there is an $k \in L^2([0,T], \mathbb{R}^+)$ such that*

$$E[\int_t^T h(F(\tau, x_\tau^1, z_\tau), F(\tau, x_\tau^2, z_\tau))d\tau | \mathcal{F}_t^z] \leq E[\int_t^T k(\tau)|x_\tau^1 - x_\tau^2| d\tau | \mathcal{F}_t^z]$$

a.s. for $0 \leq t \leq T$ and $x^1, x^2 \in \mathbb{D}(\mathbb{F}, \mathbb{R}^d)$. Then $BSDI(F, H, z)$ possesses a strong solution.

Proof. Let $\xi^z \in S_{\mathbb{F}}(H)(z)$ and let $(x_t^0)_{0 \leq t \leq T}$ be an d-dimensional $\mathbb{F}^z$-adapted process on $\mathcal{P}_{\mathbb{F}}$ such that $x_T^0 = \int_0^T \xi_t^z dt$ a.s. Similarly as in the proof of ([8], Th. 6) we define a sequence $(x^n)_{n=1}^\infty \in \mathcal{S}^2(\mathbb{F}^z, \mathbb{R}^d)$ such that

$$\begin{cases} x_s^n \in E\left[x_t^n + \int_s^t F(\tau, x_\tau^{n-1}, z_\tau)d\tau | \mathcal{F}_s^z\right] \\ x_T = \int_0^T \xi_t^z dt \end{cases} \tag{5.2}$$

a.s. for $0 \leq t \leq T$ and

$$\sup_{t \leq u \leq T} |x^{n+1} - x_u^n| \leq \sup_{t \leq u \leq T} E[\int_u^T k(\tau) \sup_{\tau \leq s \leq T} |x_s^n - x_s^{n-1}| d\tau | \mathcal{F}_u^z] \leq$$

$$\sup_{t \leq u \leq T} E[\int_t^T k(\tau) \sup_{\tau \leq s \leq T} |x_s^n - x_s^{n-1}| d\tau | \mathcal{F}_u^z]$$

a.s. for $0 \leq t \leq T$ and $n = 1, 2, \ldots$. By Doob's inequality it follows

$$E[\sup_{t \leq u \leq T} |x^{n+1} - x_u^n|^2] \leq 4E\left(\int_t^T k(\tau) \sup_{\tau \leq s \leq T} |x_s^n - x_s^{n-1}| d\tau\right)^2$$

for $0 \leq t \leq T$ and $n = 1, 2, \ldots$. Hence it follows

$$E \sup_{t \leq u \leq T} |x^{n+1} - x_u^n|^2 \leq \frac{(4ET)^n \mathbb{L}^{n-1}}{n!} \left(\int_t^T k^2(\tau)d\tau\right)^n$$

for $n = 1, 2, \ldots$, where $\mathbb{L} = 4T \int_0^T m^2(t)dt$ with $m \in L^2([0,T], \mathbb{R}^+)$ such that $\max(h(F(t,x,z), \{0\}), h(H(t,z), \{0\})) \leq m(t)$ for $(x,z) \in \mathbb{R}^{d+m}$ and a.e. $t \in [0,T]$. Then $E \sup_{0 \leq t \leq T} |x_t^k - x_t^n|^2 \to 0$ as $k, n \to \infty$. Therefore there is a process $(x_t)_{0 \leq t \leq T} \in \mathcal{S}^2(\mathbb{F}^z, \mathbb{R}^d)$ such that $E \sup_{0 \leq t \leq T} |x_t^n - x_t|^2 \to 0$ as $n \to \infty$. Similarly as in the proof of ([8], Th. 6) we obtain

$$\begin{cases} x_s \in E\left[x_t + \int_s^t F(\tau, x_\tau, z_\tau)d\tau | \mathcal{F}_s^z\right] \\ x_T \in \int_0^T H(t, z_t)dt \end{cases} \tag{5.3}$$

a.s. for $0 \leq t \leq T$. □

Immediately from ([1], Th. 4.1) and the Caratheodory selection theorem (see [12], Th. 2) the following existence theorem follows.

Theorem 5.4. *Let $F : [0,T] \times \mathbb{R}^d \times \mathbb{R}^m \to Cl(\mathbb{R}^m)$ and $H : [0,T] \times \mathbb{R}^m \to Cl(\mathbb{R}^m)$ be measurable and uniformly integrable bounded such that $F(t,\cdot,\cdot)$ and $H(t,\cdot)$ are l.s.c. for a.e. fixed $t \in [0,T]$. Then for every probability measure μ on $\mathcal{D}(\mathbb{R}^m)$ there is a weak solution to $BSDI(F,H,\mu)$.*

Proof. By virtue of ([12], Th. 10) there are Carathéodory selectors f and h of $co\ F$ and $co\ H$, respectively. Let $g : [0,T] \times D(\mathbb{R}^d) \times D(\mathbb{R}^m) \to \mathbb{R}^d$ and $l : D(\mathbb{R}^m) \to \mathbb{R}^d$ be defined by $g(t,x,z) = f(t,x_t,z_t)$ and $l(z) = \int_0^T h(t,z_t)dt$ for a.e. $t \in [0,T]$, $x \in D(\mathbb{R}^d)$ and $z \in D(\mathbb{R}^m)$, respectively. It is easy to see that g and l satisfy the assumptions of ([1], Th. 4.1). Therefore for every probability measure μ on $\mathcal{D}(\mathbb{R}^m)$ there is a weak solution $(\Omega, \mathcal{F}, \mathbb{F}^{xz}, Q, x, z)$ to the $BSDE(l,g,\mu)$. It is easy to verify that

$$\left\{ \begin{array}{l} x_s \in E_Q\left[x_t + \int_s^t F(\tau, x_\tau, z_\tau)d\tau | \mathcal{F}_s^{xz}\right] \\ x_T \in \int_0^T H(t,z_t)dt \end{array} \right. \tag{5.4}$$

Q-a.s. for $0 \le t \le T$. ☐

In a similar way we also obtain.

Theorem 5.5. *Let $F : [0,T] \times \mathbb{R}^d \times \mathbb{R}^m \to Cl(\mathbb{R}^m)$ and $H : [0,T] \times \mathbb{R}^m \to Cl(\mathbb{R}^m)$ be measurable and uniformly integrable and let μ be a probability measure on $\mathcal{D}(\mathbb{R}^m)$. If F and H are $\mathcal{A}$-l.s.c. then $BSDI(F,H,\mu)$ possesses a weak solution.*

Proof. Let $S(F)$ and $S(H)$ be set-valued mappings defined by $S(F)(x,z) = \{g \in L([0,T],\mathbb{R}^d) : g(t) \in F(t,x,z);\ a.e.\ 0 \le t \le T\}$ and $S(H)(z) = \{h \in L([0,T],\mathbb{R}^d) : h(t) \in H(t,z);\ a.e.\ 0 \le t \le T\}$ for fixed $x \in \mathbb{R}^d$ and $z \in \mathbb{R}^m$, respectively. It can be proved (see [6], Th. III. 2.1 and Lemma III.2.9) that $S(F)$ and $S(H)$ have closed bounded and decomposable values at $L([0,T],\mathbb{R}^d)$ and are l.s.c. as mappings from $\mathbb{R}^d \times \mathbb{R}^m$ and $\mathbb{R}^m$, respectively to a complete metric space $Cl(L([0,T],\mathbb{R}^d)), d)$. Therefore, by Bressan-Colombo-Fryszkowski continuous selection theorem (see [2], Th. 42) there are $\mathcal{A}$-continuous functions $f : \mathbb{R}^d \times \mathbb{R}^m \to L([0,T],\mathbb{R}^d)$ and $h : \mathbb{R}^m \to L([0,T],\mathbb{R}^d)$ such that $f(x,z)(t) \in F(t,x,z)$ and $h(z)(t) \in H(t,z)$ for a.e. $t \in [0,T]$, $x \in \mathbb{R}^d$ and $z \in \mathbb{R}^m$, respectively. Let $g(t,x,z) = f(x_t,z_t)(t)$ and $l(z) = \int_0^T h(z_t)(t)dt$ for a.e. $t \in [0,T]$, $x \in D(\mathbb{R}^d)$ and $z \in D(\mathbb{R}^m)$, respectively. Let us observe that for every measurable functions $\varphi : [0,T] \to \mathbb{R}^d$ and $\psi : [0,T] \to \mathbb{R}^m$ a function $[0,T] \ni t \to f(\varphi(t),\psi(t))(t) \in \mathbb{R}^d$ is measurable. For fixed $u \in D(\mathbb{R}^r)$, $\delta > 0$ and $k \ge 1$ we define

$$\varphi_\delta^k(t,u) = \frac{1}{\delta}\int_t^{T\wedge(t+\delta)} \pi_s^r(u)\left((k+1-|\pi_s^r(u)|)^+ \wedge 1\right)ds$$

for $t \in [0,T]$. It is clear that a function $\varphi_\delta^k(\cdot,u)$ is measurable. Furhermore a mapping $D(\mathbb{R}^r) \ni u \to \varphi_\delta^k(t,u) \in \mathbb{R}^r$ is continuous in the Meyer-Zheng topology and hence $\mathcal{D}(\mathbb{R}^r)$-measurable. This yields that the coordinete mapping π_t^r, as a poinwise limit of $\varphi_\delta^k(t,\cdot)$ by $k \to \infty$ and $\delta \to 0$, is $\mathcal{D}(\mathbb{R}^r)$-nmeasurable for all $t \in [0,T]$. Finally, from the definition of $\mathcal{D}(\mathbb{R}^r)$, it follows that also $\pi_T^r = \lim_{\delta\to 0}\pi_{T-\delta}^r$ is $\mathcal{D}(\mathbb{R}^r)$-measurable. In a similar way we can verify that for every fixed $t \in [0,T]$ a coordinate mapping π_t^r is $\mathcal{D}_t^T$-measurable. Hence it follows that a mapping g defined above is such that $g(\cdot,x,z)$ is measurable and $g(t,x,\cdot)$ is $\mathcal{D}_t^T$-measurable for fixed $(x,z) \in D(\mathbb{R}^d)\times D(\mathbb{R}^m)$ and $(t,z) \in [0,T]\times D(\mathbb{R}^m)$, respectively, because $g(t,x,z) = f(t,x_t,z_t) = f(t,\pi_t^d(x),\pi_t^m(z))$. Finally, it can be verified that a function $D(\mathbb{R}^d)\times D(\mathbb{R}^m) \ni (x,z) \to g(\cdot,x,z) \in L([0,T],\mathbb{R}^d)$ is $\mathcal{A}$-continuous on $(D(\mathbb{R}^d)\times D(\mathbb{R}^m), r)$, where $r((x,z),(u,v)) = \max(\rho(x,u),\rho(z,v))$ for $(x,z),(u,v) \in (D(\mathbb{R}^d)\times D(\mathbb{R}^m), r)$. Indeed, let $(x^n,z^n)_{n=1}^\infty$ be a sequence of $D(\mathbb{R}^d)\times D(\mathbb{R}^m), r)$ converging in the r-metric topology to $(x,z) \in D(\mathbb{R}^d)\times D(\mathbb{R}^m)$. Taking an arbitrary subsequence $(x^{n_k},z^{n_k})_{n=1}^\infty$ of $(x^n,z^n)_{n=1}^\infty$ we can select its subsequence, denoting again by $(x^{n_k},z^{n_k})_{n=1}^\infty$, such that $x_t^{n_k} \to x_t$ and $z_t^{n_k} \to z_t$ for a.e. $t \in [0,T]$ as $k \to \infty$. By the $\mathcal{A}$-continuity of f hence it follows $\int_0^T |f(x_t^{n_k},z_t^{n_k})(t) - f(x_t,z_t)(t)|dt \to 0$ as $k \to \infty$. Therefore $\int_0^T |f(x_t^n,z_t^n)(t) - f(x_t,z_t)(t)|dt \to 0$ as $n \to \infty$. Quite similar we can verify that a function $D(\mathbb{R}^m) \ni z \to l(z) = \int_0^T h(z_t)(t)dt \in \mathbb{R}^d$ is $\mathcal{D}(\mathbb{R}^m)$-measurable. Now, immediately from ([1], Th. 4.1) it follows that $BSDE(l,g,\mu)$ possesses at least one weak solution $(\mathcal{P}_{\mathbb{F}},x,z)$. By the properties of mappings g and l it follows that $(\mathcal{P}_{\mathbb{F}},x,z)$ is a weak solution to $BSDI(F,H,\mu)$. □

6. WEAK COMPACTNESS OF WEAK SOLUTIONS SET TO (F,H,μ)

We shall show that if F and H satisfies the assumptions of Theorem 5.4 and are $\mathcal{A}$-continuous with the respect to their last arguments then for every weakly compact set Λ of probability measures on $\mathcal{D}(\mathbb{R}^m)$ the set $\mathcal{X}(F,H,\Lambda)$ of all weak solutions to $BSDI(F,H,\mu)$ with $\mu \in \Lambda$ is weakly compact with respect to the Meyer-Zheng topology. We begin with the following result.

Theorem 6.1. *Let $F : [0,T]\times\mathbb{R}^d\times\mathbb{R}^m \to \mathbb{R}^d$ and $H : [0,T]\times\mathbb{R}^m \to \mathbb{R}^d$ be measurable, uniformly integrable bounded and $\mathcal{A}$-continuous with respect to their last two or last variables, respectively. For every nonempty weakly closed set Λ of probability measures on $\mathcal{D}(\mathbb{R}^m)$ the set $\mathcal{X}(F,H,\Lambda)$ is nonempty and weakly closed in the Meyer-Zheng topology.*

Proof. By virtue of Theorem 5.5 we have $\mathcal{X}(F,H,\Lambda) \neq \emptyset$. Let $(\mathcal{P}_{\mathbb{F}^n}^n, X^n, Y^n)_{n=1}^\infty$ be a sequence of $\mathcal{X}(F,H,\Lambda)$ weakly converging in the Meyer-Zheng topology, where $\mathcal{P}_{\mathbb{F}^n}^n = (\Omega^n,\mathcal{F}^n,P^n,\mathbb{F}^n)$ with $\mathbb{F}^n = (\mathcal{F}_t^n)_{0\le t\le T}$. Then there is a probability measure Q on $(D(\mathbb{R}^d)\times D(\mathbb{R}^m),\mathcal{D}(\mathbb{R}^{d+m}))$ such that a sequence $(Q^n)_{n=1}^\infty$ of the distributions of (X^n,Y^n) with respect to P^n converges weakly to Q in the Meyer-Zheng topology. For every $n = 1,2,\ldots$ we have: $P^n(Y^n)^{-1} = \mu_n$, $\sigma(p,X_s^n) \le \sigma(p,E^n[X_t^n + \int_0^t F(\tau,X_\tau^n,Y_\tau^n)d\tau|\mathcal{F}_s^n])$ and $\sigma(p,X_s^n) \le \sigma(p,\int_0^T H(\tau,Y_\tau^n)d\tau)$ P^n-a.s.

for every $0 \leq s \leq t \leq T$ and $p \in \mathbb{R}^d$. Similarly as above, let π^x and π^y denote the projections on $, D(\mathbb{R}^{d+m})$ defined by $\pi^x(x,y) = x$ and $\pi^y(x,y) = y$ for $(x,y) \in D(\mathbb{R}^{d+m})$. Similarly, by π_t^x and π_t^y we denote the coordinate mappings defined by $\pi_t^x(x,y) = x_t$ and $\pi_t^y(x,y) = y_t$ for $(x,y) \in D(\mathbb{R}^{d+m})$ and $t \in [0,T]$. Finally, by $\mathbb{F}^{xy}$ we denote the smallest filtration satisfying the usual conditions such that $(\pi_t^x, \pi_t^y)_{0 \leq t \leq T}$ is $\mathbb{F}^{xy}$ - adapted. In what follows, we shall denote processes $(\pi_t^x)_{0 \leq t \leq T}$ and $(\pi_t^y)_{0 \leq t \leq T}$ by $X = (X_t)_{0 \leq t \leq T}$ and $Y = (Y_t)_{0 \leq t \leq T}$, respectively. We shall show that $(\mathcal{P}_{\mathbb{F}^{xy}}, X, Y)$ is a solution to $BSDI(F,H,\mu)$, where $\mu \in \Lambda$ is such that $\mu_n \to \mu$ weakly as $n \to \infty$ and $\mathcal{P}_{\mathbb{F}^{xy}} = (\Omega, \mathcal{F}, Q, \mathbb{F}^{xy})$ with $\Omega = D(\mathbb{R}^{d+m})$ and $\mathcal{F} = \mathcal{D}^Q(\mathbb{R}^{d+m})$, where $\mathcal{D}^Q(\mathbb{R}^{d+m})$ denotes the completion of $\mathcal{D}(\mathbb{R}^{d+m})$ with respect to Q. Let $\Phi : D(\mathbb{R}^{d+m}) \to \mathbb{R}$ be bounded and continuous with respect to the Meyer-Zheng topology. For every $t \in [0,T]$ we put $\varphi(t,x,y) = \Phi(x^t, y^t)$ for $(x,y) \in D(\mathbb{R}^{d+m})$, where $x^t(s) = x_s \mathbf{1}_{[0,t]}(s)$ and $y^t(s) = y_s \mathbf{1}_{[0,t]}(s)$ for $s \in [0,t]$. It is clear that for every $t \in [0,T]$, $\varphi(t,\cdot,\cdot)$ is bounded and continuous in the Meyer-Zheng topology on $D(\mathbb{R}^{d+m})$. Furthermore, $\varphi(t,\cdot,\cdot)$ is $\mathcal{F}_t^{xy}$-measurable. Therefore

$$(6.1) \qquad E_{Q^n}\sigma(p, \varphi(s,X,Y)X_s) \leq E_{Q^n}\sigma(p, E_{Q^n}[\varphi(s,X,Y)(X_t + \int_s^t F(\tau, X_\tau, Y_\tau)d\tau)|\mathcal{F}_s^{xy}]) = E_{Q^n}\sigma\left(p, \varphi(s,X,Y)(X_t + \int_s^t F(\tau, X_\tau, Y_\tau)d\tau)\right)$$

and

$$(6.2) \qquad E_{Q^n}\sigma(p, \varphi(s,X,Y)X_T) \leq E_{Q^n}\sigma\left(p, \varphi(s,X,Y)\int_0^T H(\tau, Y_\tau)d\tau\right)$$

for $n = 1,2,\ldots$ and $0 \leq s \leq t \leq T$. Similarly as in the proof of ([1], Th. 4.1) we can define $X_t^k = X_t\left((k+1-|X_t|)^+\right)$ for $k \geq 0$ and verify that for every $\varepsilon > 0$ there is a $k(\varepsilon)$ such that

(i) $\sup_{0 \leq t \leq T} E_{Q^n}|X_t^k - X_t| \leq \varepsilon$ for $n \geq 1$ and $k \geq k(\varepsilon)$
(ii) $\sup_{0 \leq t \leq T} E_Q|X_t^k - X_t| \leq \varepsilon$ for $k \geq k(\varepsilon)$.

Furthermore for given $\varepsilon > 0$ and $k > 0$ there is a $\delta(\varepsilon, k) > 0$ (depending on t) such that

(iii) $E_Q\left|\frac{1}{\delta}\int_t^{t+\delta} X_\tau^k d\tau - X_t^k\right| \leq \varepsilon$ for $\delta \leq \delta(\varepsilon,k)$
(iv) $\sup_{0 \leq t \leq T}\left|E_{Q^n}\left[\frac{1}{\delta}\int_t^{t+\delta} X_\tau^k d\tau - X_t^k\right]\right| \leq \varepsilon$ for $\delta \leq \varepsilon / \int_0^T m(t)dt$,

where $m \in L([0,T], \mathbb{R}^+)$ is such that $\max\{h(F(t,x,y),\{0\}), h(H(t,y),\{0\})\} \leq m(t)$ for $(x,y) \in \mathbb{R}^d \times \mathbb{R}^m$ and a.e. $t \in [0,T]$. For every $p \in \mathbb{R}^d$ and fixed $0 \leq s \leq t \leq T$ one has

$$E_Q\sigma\left(p, \varphi(s,X,Y)X_s\right) - E_Q\sigma\left(p, \varphi(s,X,Y)(X_t + \int_s^t F(\tau, X_\tau, Y_\tau)d\tau)\right) =$$

$$E_Q\left[\sigma\left(p, \varphi(s,X,Y)X_s\right) - \sigma\left(p, \varphi(s,X,Y)X_s^k\right)\right] +$$

$$E_Q\left[\sigma\left(p,\varphi(s,X,Y)X_s^k\right)-\sigma\left(p,\varphi(s,X,Y)\frac{1}{\delta}\int_s^{s+\delta}X_s^k\right)\right]+$$
$$E_Q\left[\sigma\left(p,\varphi(s,X,Y)\frac{1}{\delta}\int_s^{s+\delta}X_s^k\right)\right]-E_{Q^n}\left[\sigma\left(p,\varphi(s,X,Y)\frac{1}{\delta}\int_s^{s+\delta}X_s^k\right)\right]+$$
$$E_{Q^n}\left[\sigma\left(p,\varphi(s,X,Y)\frac{1}{\delta}\int_s^{s+\delta}X_s^k\right)-\sigma\left(p,\varphi(s,X,Y)X_s^k\right)\right]+$$
$$E_{Q^n}\left[\sigma\left(p,\varphi(s,X,Y)X_s^k\right)-\sigma\left(p,\varphi(s,X,Y)X_s\right)\right]+$$
$$E_{Q^n}\left[\sigma\left(p,\varphi(s,X,Y)X_s\right)\right]-E_{Q^n}\left[\sigma\left(p,\varphi(s,X,Y)\right)\left(X_t+\int_s^t F(\tau,X_\tau,Y_\tau)d\tau\right)\right]+$$
$$E_{Q^n}\left[\sigma\left(p,\varphi(s,X,Y)\right)\left(X_t+\int_s^t F(\tau,X_\tau,Y_\tau)d\tau\right)\right]-$$
$$E_{Q^n}\left[\sigma\left(p,\varphi(s,X,Y)\right)\left(X_t^k+\int_s^t F(\tau,X_\tau,Y_\tau)d\tau\right)\right]+$$
$$E_{Q^n}\left[\sigma\left(p,\varphi(s,X,Y)\right)\left(X_t^k+\int_s^t F(\tau,X_\tau,Y_\tau)d\tau\right)\right]-$$
$$E_{Q^n}\left[\sigma\left(p,\varphi(s,X,Y)\right)\left(\frac{1}{\delta}\int_s^{s+\delta}X_\tau^k d\tau+\int_s^t F(\tau,X_\tau,Y_\tau)d\tau\right)\right]+$$
$$E_{Q^n}\left[\sigma\left(p,\varphi(s,X,Y)\right)\left(\frac{1}{\delta}\int_s^{s+\delta}X_\tau^k d\tau+\int_s^t F(\tau,X_\tau,Y_\tau)d\tau\right)\right]-$$
$$E_{Q}\left[\sigma\left(p,\varphi(s,X,Y)\right)\left(\frac{1}{\delta}\int_s^{s+\delta}X_\tau^k d\tau+\int_s^t F(\tau,X_\tau,Y_\tau)d\tau\right)\right]+$$
$$E_{Q}\left[\sigma\left(p,\varphi(s,X,Y)\right)\left(\frac{1}{\delta}\int_s^{s+\delta}X_\tau^k d\tau+\int_s^t F(\tau,X_\tau,Y_\tau)d\tau\right)\right]-$$
$$E_{Q}\left[\sigma\left(p,\varphi(s,X,Y)\right)\left(X_t^k+\int_s^t F(\tau,X_\tau,Y_\tau)d\tau\right)\right]+$$
$$E_{Q}\left[\sigma\left(p,\varphi(s,X,Y)\right)\left(X_t^k+\int_s^t F(\tau,X_\tau,Y_\tau)d\tau\right)\right]-$$
$$E_{Q}\left[\sigma\left(p,\varphi(s,X,Y)\right)\left(X_t+\int_s^t F(\tau,X_\tau,Y_\tau)d\tau\right)\right].$$

Hence it follows

$$E_Q\sigma\left(p,\varphi(s,X,Y)X_s\right)\le E_Q\sigma\left(p,\varphi(s,X,Y)(X_t+\int_s^t F(\tau,X_\tau,Y_\tau)d\tau)\right)$$

for every $p\in\mathbb{R}^d$ $0\le s\le t\le T$. Therefore, for every fixed $0\le s\le t\le T$ one has

$$E_Q\varphi(s,X,Y)X_s\in E_Q\varphi(s,X,Y)(X_t+\int_s^t F(\tau,X_\tau,Y_\tau)d\tau)$$

Let f^{st} be for every fixed $0\le s\le t\le T$ an $dt\times Q$-measurable and $\mathbb{F}^{xy}$-adapted selector of a set-valued mapping G defined by $G(\tau,\omega)=F(\tau,X_\tau(\omega),Y_\tau(\omega))$ for $(\tau,\omega)\in[s,t]\times\Omega$ such that

$$E_Q\varphi(s,X,Y)X_s=E_Q\varphi(s,X,Y)\left(X_t+\int_s^t f_\tau^{st}d\tau\right).$$

By the monotone class theorem this equality can be extended to

$$E_Q\varphi(s,X,Y)X_s = E_Q\varphi(s,X,Y)\left(X_t + \int_s^t f_\tau^{st}d\tau\right)$$

for every bounded and measurable functions $\Phi : D(\mathbb{R}^{d+m}) \to \mathbb{R}$, where again as above we put $\varphi(s,X,Y) = \Phi(X^s,Y^s)$ with X^s and Y^s defined such as above. But $\varphi(s,X,Y)$ runs over all bounded and $\mathcal{F}_{s-}^{xy} = \sigma(\{X_u,Y_u : u < s\})$-measurable functions. Therefore, similarly as in ([1], Th4.1), we can conclude that

$$E_Q\left[X_s - X_t - \int_s^t f_\tau^{st}d\tau|\mathcal{F}_{s-}^{xy}\right] = 0$$

Q-a.s. for every fixed $0 \leq s \leq t \leq T$ or equivalently,

$$E_Q\left[X_s - X_t - \int_s^t f_\tau^{st}d\tau|\mathcal{F}_u^{xy}\right] = 0$$

Q-a.s. for every $0 \leq u < s \leq t \leq T$. Hence, similarly as in ([1], Th. 4.1) it follows that

$$x_s = E_Q\left[X_t + \int_s^t f_\tau^{st}d\tau|\mathcal{F}_s^{xy}\right]$$

Q-a.s. for every $0 \leq s \leq t \leq T$, which implies that

$$x_s \in E_Q\left[X_t + \int_s^t F(\tau,X_\tau,Y_\tau)d\tau|\mathcal{F}_s^{xy}\right]$$

Q-a.s. for every $0 \leq s \leq t \leq T$. Quite similar, by the inequality (5.2) it follows that $X_T \in \int_0^T H(t,Y_t)dt$, Q-a.s. Similarly as in the proof of ([1], Th. 4.1) we can verify that every $\mathbb{F}^Y$-martingale is also $\mathbb{F}^{xy}$-martingale on $(\Omega,\mathcal{F},Q)$. Finally, let us observe that for every $A \in \mathcal{D}(\mathbb{R}^m)$ one has $\mu_n(A) = P_{Y^n}^n(A) = Q^n(D(\mathbb{R}^d)\times A)$ for $n \geq 1$. Therefore, $\mu(A) = Q(D(\mathbb{R}^d)\times A)$ for every $A \in \mathcal{D}(\mathbb{R}^m)$. Then $(\Omega,\mathcal{F},Q,\mathbb{F}^{xy},X,Y)$ is a weak solution to $BSDI(F,H,\mu)$. □

Now we can prove the main result dealing with weak compactness of the set $\mathcal{X}(F,H,\Lambda)$ of all weak solutions to $BSDI(F,H,\mu)$ with $\mu \in \Lambda$.

Theorem 6.2. *Let F, H and Λ satisfy the assumptions of Theorem 6.1. If Λ is weakly compact then the set $\mathcal{X}(F,H,\Lambda)$ is nonempty and weakly compact with respect to the Meyer-Zheng topology.*

Proof. Let $(\mathcal{P}_{\mathbb{F}^n}^n, X^n, Y^n)$ be an arbitrarily taken sequence of $\mathcal{X}(F,H,\Lambda)$ with $\mathcal{P}_{\mathbb{F}^n}^n = (\Omega^n,\mathcal{F}^n,\mathbb{F}^n,P^n)$, where $\mathbb{F}^n = (\mathcal{F}_t^n)_{0\leq t\leq T}$ for $n = 1,2,\ldots$. There is a sequence $(\mu_n)_{n=1}^\infty$ of Λ such that $(\mathcal{P}_{\mathbb{F}^n}^n, X^n, Y^n)$ is a weak solution to $BSDI(F,H,\mu_n)$ for $n = 1,2,\ldots$. By the compactness of Λ there is a subsequence $(\mu_{n_k})_{k=1}^\infty$ of $(\mu_n)_{n=1}^\infty$ and $\mu \in \Lambda$ such that $(\mu_{n_k})_{k=1}^\infty$ converges weakly to μ as $k \to \infty$. Consider now a subsequence $(\mathcal{P}_{\mathbb{F}^{n_k}}^{n_k}, X^{n_k}, Y^{n_k})$ of $(\mathcal{P}_{\mathbb{F}^n}^n, X^n, Y^n)$ and let $Q^{n_k} = P_{(X^{n_k},Y^{n_k})}^k$ be the distribution of (X^{n_k},Y^{n_k}) on $\mathcal{D}(\mathbb{R}^{d+m})$ defined with respect to the probability maesure P^{n_k} for $k = 1, 2, \dots$. Similarly as in the proof of ([1], Lemma 4.3) we can verify that there exist a subsequence, denoting still by $(Q^{n_k})_{k=1}^{\infty}$, of $(Q^{n_k})_{k=1}^{\infty}$ and a probability measure Q on $\mathcal{D}(\mathbb{R}^{d+m})$ such that $(Q^{n_k})_{k=1}^{\infty}$, converges weakly in the Meyer-Zheng topology to Q as $k \to \infty$. Moreover $Q(D(\mathbb{R}^d) \times A) = \mu(A)$ for $A \in \mathcal{D}(\mathbb{R}^m)$. Hence, similarly as in the proof of Theorem 6.1, it follows the existence of a weak solution $(\mathcal{P}_{\mathbb{F}}, X, Y)$ to $BSDI(F, H, \mu)$ such that $PY^{-1} = \mu$ which proves that $\mathcal{X}(F, H, \Lambda)$ is weakly compact with respect to the weak convergence of the distributions in the Meyer-Zheng topology. □

In a similar way we can get the following general theorem.

Theorem 6.3. *Let* $F : [0, T] \times D(\mathbb{R}^d) \times D(\mathbb{R}^m) \to Cl(\mathbb{R}^d)$ *and* $H : [0, T] \times D(\mathbb{R}^m) \to Cl(\mathbb{R}^d)$ *be measurable, uniformly integrably bounded,* $\mathcal{A}$*-continuous with respect to their last two and last variables, respectively and such that* $F(t, \cdot, \cdot)$ *and* $H(t, \cdot)$ *are* $\mathcal{D}_t(\mathbb{R}^{d+m})$*-measurable for every fixed* $t \in [0, T]$ *and* $F(t, \cdot, y)$ *is* $\mathcal{D}_t^T(\mathbb{R}^d)$*-measurable for every fixed* $(t, y) \in [0, T] \times D(\mathbb{R}^m)$. *Then for every nonempty weakly compact set* Λ *of probability measures on* $\mathcal{D}(\mathbb{R}^m)$ *the set* $\mathcal{X}(F, H, \Lambda)$ *is nonempty and weakly compact with respect to the Meyer-Zheng topology.*

Proof. It is enough only to verify that $\mathcal{X}(F, H, \Lambda) \neq \emptyset$. The weak compactness of $\mathcal{X}(F, H, \Lambda)$ can be verified similarly as in the proof of Theorem 6.2. Let $\mu \in \Lambda$ and $(\widetilde{\Omega}, \widetilde{\mathcal{F}}, \widetilde{P}) = (D^m, \mathcal{D}^{\mu}(\mathbb{R}^m), \mu)$, where $\mathcal{D}^{\mu}(\mathbb{R}^m)$ denotes the completion of $\mathcal{D}(\mathbb{R}^m)$ with respect to μ. Denote by $\mathbb{F}^{\widetilde{Y}}$ the smallest of all filtrations $\mathbb{F}^{\widetilde{Y}} = (\mathcal{F}_t^{\widetilde{Y}})_{0 \le t \le T}$ such that the process $\widetilde{Y}$ is $\mathbb{F}^{\widetilde{Y}}$ -adapted and $\mathbb{F}^{\widetilde{Y}}$ satisfies the usual conditions. Let

$$F_n(x, y) = \begin{cases} F(t + \frac{1}{n}, x, y) \quad for \quad t \in [0, T - \frac{1}{n}) \\ \{0\} \quad for \quad t \in [T - \frac{1}{n}, T] \end{cases} \tag{6.3}$$

for $n \geq 1$ and $(x, y) \in D(\mathbb{R}^d) \times D(\mathbb{R}^m)$ and let us consider $BSDI(F_n, H, \mu)$ on $(\widetilde{\Omega}, \widetilde{\mathcal{F}}, \widetilde{P})$ with a driving process $\widetilde{Y}$. We define for every $n \geq 1$ a strong solution X_n to $BSDI(F_n, H, \mu)$ with a driving process $\widetilde{Y}$. We construct such solution begining with the interval $[T - 1/n, T]$. Let Φ be an $\mathbb{F}^{\widetilde{Y}}$-martingale on $(\widetilde{\Omega}, \widetilde{\mathcal{F}}, \widetilde{P})$ such that $\Phi_T \in \int_0^T H(t, \widetilde{Y}_t)dt$, $\widetilde{P}$-a.s. To define such Φ it is enough to select an $\widetilde{\mathcal{F}}_T^{\widetilde{Y}}$-measurable random variable $\xi \in \int_0^T H(t, \widetilde{Y}_t)dt$ and take $\Phi_t = \widetilde{E}[\xi|\widetilde{\mathcal{F}}_t^{\widetilde{Y}}]$ for $t \in [0, T]$. Let $X^n = \mathbb{1}_{[T-1/n, T]}\Phi$. For every $T - 1/n \leq s \leq t \leq T$ one has $X_s^n = \Phi_s = \widetilde{E}[\phi_t|\widetilde{\mathcal{F}}_s] = \widetilde{E}[X_t^n|\mathcal{F}_s^{\widetilde{Y}}]$, $\widetilde{P}$-a.s. Then $X_s^n = \widetilde{E}[X_t^n + \int_s^t 0 d\tau|\mathcal{F}_s^{\widetilde{Y}}] \in \widetilde{E}[X_t^n + \int_s^t F_n(\tau, X^n, \widetilde{Y})d\tau|\mathcal{F}_s^{\widetilde{Y}}]$, $\widetilde{P}$-a.s for $T - 1/n \leq s \leq t \leq T$. For every $T - 2/n \leq s \leq t \leq T - 1/n$ we have $\int_s^t F_n(\tau, X^n, \widetilde{Y})d\tau = \int_s^t F(\tau + \frac{1}{n}, X^n, \widetilde{Y})d\tau = \int_{s+1/n}^{t+1/n} F(\tau, \Phi, \widetilde{Y})d\tau = \int_s^t F_n(\tau, \Phi, \widetilde{Y})d\tau$ beacause $F(t, \cdot, \widetilde{Y})$ is $\mathcal{D}_t^T(\mathbb{R}^d)$-measurable. Let $(g_t^n)_{T-2/n \leq t \leq T-1/n}$ be an $\mathbb{F}^{\widetilde{Y}}$-adapted and integrable process on $(\widetilde{\Omega}, \widetilde{\mathcal{F}}, \widetilde{P})$

such that $\int_s^t g_\tau^n d\tau \in \int_s^t F_n(\tau,\Phi,\widetilde{Y})d\tau$, $\widetilde{P}$-a.s. and put $f^n = \mathbb{1}_{[T-2/n,T-1/n]}g^n$. We can redefine process X^n by taking $X_t^n = \widetilde{E}[\Phi_T + \int_t^T f_\tau^n d\tau|\mathcal{F}_t^{\widetilde{Y}}]$, $\widetilde{P}$-a.s. for $t \in [T-2/n,T]$ and $X_t^n = 0$, $\widetilde{P}$-a.s. for $t \in [0,T-2/n)$. We obtain $X_s^n = \widetilde{E}[\Phi_T + \int_s^T |\mathcal{F}_s^{\widetilde{Y}}] = \widetilde{E}[\Phi_T + \int_t^T f_\tau^n d\tau|\mathcal{F}_s^{\widetilde{Y}}] + \widetilde{E}[\int_s^t f_\tau^n d\tau|\mathcal{F}_s^{\widetilde{Y}}] = \widetilde{E}[\widetilde{E}[\Phi_T + \int_t^T f_\tau^n d\tau|\mathcal{F}_t^{\widetilde{Y}}]|\mathcal{F}_s^{\widetilde{Y}}]] + \widetilde{E}[\int_s^t f_\tau^n d\tau|\mathcal{F}_s^{\widetilde{Y}}]] = \widetilde{E}[X_t^n + \int_s^t f_\tau^n d\tau|\mathcal{F}_s^{\widetilde{Y}}] \in \widetilde{E}[X_t^n + \int_s^t F^n(\tau,X^n,\widetilde{Y})d\tau|\mathcal{F}_s^{\widetilde{Y}}]$, $\widetilde{P}$-a.s. for $[T-2/n \le s \le t \le T-1/n]$. For $s \in [T-2/n,T-1/n]$ and $t \in [T-1/n,T]$ we have $X_s^n = \widetilde{E}[\Phi_T + \int_s^T f_\tau^n|\mathcal{F}_s^{\widetilde{Y}}]$, and $X_t^n = \widetilde{E}[\Phi_T|\mathcal{F}_t^{\widetilde{Y}}]$ because $\int_t^T f_\tau^n d\tau = 0$, $\widetilde{P}$-a.s. Therefore, $X_s^n = \widetilde{E}[\widetilde{E}[\Phi_T|\mathcal{F}_t^{\widetilde{Y}}]|\mathcal{F}_s^{\widetilde{Y}}]+\widetilde{E}[\int_s^t f_\tau^n d\tau|\mathcal{F}_s^{\widetilde{Y}}] = \widetilde{E}[X_t^n+\int_s^t f_\tau^n d\tau|\mathcal{F}_s^{\widetilde{Y}}] \in \widetilde{E}[X_t^n + \int_s^t F^n(\tau,X^n,\widetilde{Y})d\tau|\mathcal{F}_s^{\widetilde{Y}}]$, $\widetilde{P}$-a.s. In a similar way we can redifine the process X^n on the whole interval $[0,T]$ in such a way that $X_s^n \in \widetilde{E}[X_t^n + \int_s^t F^n(\tau,X^n,\widetilde{Y})d\tau|\mathcal{F}_s^{\widetilde{Y}}]$, $\widetilde{P}$-a.s. for $0 \le s \le t \le T$. It is clear that F_n is measurable, uniformly integrable bounded, $\mathcal{A}$-continuous with respect to its last variables and such that $F(t,\cdot,\cdot)$ is $\mathcal{D}_t(\mathbb{R}^{d+m})$-measurable for fixed $t \in [0,T]$.

Denote by Q^n the ditribution of of $(X^n,\widetilde{Y})$ on $\mathcal{D}(\mathbb{R}^{d+m})$ with respect to $\widetilde{P}$. Similarly as in the proof of ([1], Lemma 4.3) we can verify that there is a subsequence $(Q^{n_k})_{k=1}^\infty$ of $(Q^n)_{n=1}^\infty$ and a probability measure Q on $\mathcal{D}(\mathbb{R}^{d+m})$ converging weakly in the Meyer-Zheng topology to Q and such that $Q(D(\mathbb{R}^d)\times A) = \mu(A)$ for every $A \in \mathcal{D}(\mathbb{R}^m)$ as $k \to \infty$. Let X, Y, $(\Omega,\mathcal{F},Q,\mathbb{F}^{xy})$ and φ be such as in the proof of Theorem 6.1, where Q is a probability measure defined above. By the definition of Q^n we have

$$E_{Q^n}\sigma\left(p,\varphi(s,X,Y)X_s\right) \le$$

$$E_{Q^n}\sigma\left(p,E_{Q^n}\left[\varphi(s,X,Y)\left(X_t + \int_s^t F_n(\tau,X,Y)d\tau)|\mathcal{F}_s^{xy}\right]\right)\right) =$$

$$E_{Q^n}\sigma\left(p,\varphi(s,X,Y)\left(X_t + \int_s^t F_n(\tau,X,Y)d\tau\right)\right)$$

and

$$E_{Q^n}\sigma\left(p,\varphi(s,X,Y)X_T\right) \le E_{Q^n}\sigma\left(p,\varphi(s,X,Y)\int_0^T H(\tau,Y)d\tau\right)$$

for $n = 1,2\ldots$ and $0 \le s \le t \le T$. Similarly as in the proof of Theorem 6.1 (see also [1], Th. 4.1) hence it follows that $(\Omega,\mathcal{F},Q,\mathbb{F}^{xy},X,Y)$ is a weak solution to $BSDI(F,H,\mu)$. □

7. EXISTENCE OF OPTIMAL WEAK SOLUTIONS

As a natural application of the main results of the paper we can consider the existence of optimal weak solutions BSDIs with respect to a given utility functional. To begin with suppose that the state of some parameters of a dynamical system at the time t is described by a pair of cádlág prosesses (X_t,Z_t) defined on a filtered probability space $\mathcal{P}_{\mathbb{F}} = (\Omega, \mathcal{F}, \mathbb{F}, P)$ with $\mathbb{F} = (\mathcal{F}_t)_{0\le t\le T}$, satisfying a system of BSDEs

$$X_t = E[X_T + \int_t^T f(s, X_s, Z_s, u_s)ds|\mathcal{F}_t]$$

depending on a control process $u = (u_t)_{0\le t\le T} \in \mathcal{U}$ and informations contained in $\mathcal{F}_t$. On the set $\mathcal{X}$ of all such pairs defined by the above BSDEs with u running over the set $\mathcal{U}$, we can define an utility functional $\mathcal{T}$ by setting

$$\mathcal{T}(X, Z) = E^{X,Z}\left[\int_0^T \Psi(X_t, Z_t)dt\right],$$

where $E^{X,Z}$ denotes the mean value operator with respect to the distribution measure $Q_{X,Z}$ on $\mathcal{D}(\mathbb{R}^{d+m})$ of a pair processes (X, Z) and $\Psi : D(\mathbb{R}^{d+m}) \to L([0, T], \mathbb{R})$ is a given $\mathcal{A}$-continuous mapping. Having given a set Λ of probability measures on $\mathcal{D}(\mathbb{R}^m)$ and a measurable and uniformly integrable bounded set-valued mapping $H : [0, T] \times \mathbb{R}^m \to Cl(\mathbb{R}^d)$ such that $H(t, \cdot)$ is continuous, we can consider a set $\mathcal{X}_f(H, \Lambda)$ of all pairs $(X, Z) \in \mathcal{X}$ such that $PZ^{-1} \in \Lambda$ and $X_T \in \int_0^T H(t, Z_t)dt$. The set $\mathcal{X}_f(H, \Lambda)$ is called an attainable set. It can be verified that $\mathcal{X}_f(H, \Lambda) = \mathcal{X}(F, H, \Lambda)$, where $\mathcal{X}(F, H, \Lambda)$ is the set of all weak solutions to $BSDI(F, F, \mu)$, with $\mu \in \Lambda$ and $F(t, x, y) = \{f(t, x, y, u) : u \in \mathcal{U}\}$. From the practical point of view it can be interested to look for a pair $(\widetilde{x}, \widetilde{z}) \in \mathcal{X}(F, H, \Lambda)$ such that $\mathcal{T}(\widetilde{x}, \widetilde{z}) = \inf\{\mathcal{T}(X, Z) : (X, Z) \in \mathcal{X}(F, H, \Lambda)\}$. We can also look for a pair $(\widetilde{x}, \widetilde{z}) \in \mathcal{X}(F, H, \Lambda)$ such that $\mathcal{T}(\widetilde{x}, \widetilde{z}) = \sup\{\mathcal{T}(X, Z) : (X, Z) \in \mathcal{X}(F, H, \Lambda)\}$. By the weak compactness of the set $\mathcal{X}(F, H, \Lambda)$ and weak continuity of a functional $\mathcal{T}$ with respect to the Meyer-Zheng topology we can obtain the existence of the above mention pair $(\widetilde{x}, \widetilde{z}) \in \mathcal{X}(F, H, \Lambda)$. To see that let us consider an arbitrary sequence $\{(X^k, Z^k)\}_{k=1}^{\infty}$ of $\mathcal{X}(F, H, \Lambda)$ weakly converging with respect to the Meyer-Zheng topology to $(X, Z) \in \mathcal{X}(F, H, \Lambda)$. Similarly as in the proofs of the above theorems we can get $\lim_{k\to\infty} \mathcal{T}(X^k, Z^k) = \mathcal{T}(X, Z)$. Then the existence of an $\mathcal{T}$-optimal weak solutions to $BSDI(F, H, \Lambda)$ can be obtained.

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