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ON A FRACTIONAL DIRICHLET PROBLEM OF HIGHER ORDER

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ABSTRACT. We consider a fractional version of some $2n$th order Dirichlet problem. In the paper a sufficient condition for the existence of solution to the aforementioned problem is proved. The proved is based on some variational methods and application of a fractional counterpart of the du Bois-Reymond lemma for the order $\alpha \in \left(n - \frac{1}{2}, n\right)$ (see [1]).

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1. INTRODUCTION

In the paper we consider the following fractional boundary problem

(1.1)
$$\begin{aligned} D^1 \Big(\ldots D^1 \Big(& D^{\alpha-(n-1)}_{b-} D^{\alpha}_{a+} x(t) - I^{n-\alpha}_{b-} F_{x_1} \left(D^{\alpha-1}_{a+} x(t), \ldots, D^{\alpha-(n-1)}_{a+} x(t), x(t), t \right) \Big) \\ & + \ldots + (-1)^{n-1} I^{n-\alpha}_{b-} F_{x_{n-1}} \left(D^{\alpha-1}_{a+} x(t), \ldots, D^{\alpha-(n-1)}_{a+} x(t), x(t), t \right) \Big) \\ & = (-1)^{n-1} F_{x_n} \left(D^{\alpha-1}_{a+} x(t), \ldots, D^{\alpha-(n-1)}_{a+} x(t), x(t), t \right) \qquad \text{for a.e. } t \in [a,b] \end{aligned}$$

(1.2) $$\left(D^i I^{n-\alpha}_{a+} x \right)(a) = 0 \text{ and } \left(I^{a-i} D^{\alpha}_{a+} x \right)(b) = 0 \text{ for } i = 0, \ldots, n-1$$

where $\alpha \in \left(n - \frac{1}{2}, n\right)$, $n = 2, \ldots$, $F = F(x_1, \ldots x_n, t) : (\mathbb{R}^m)^n \times [a,b] \to \mathbb{R}^m$ and $D^{\beta}_{a+} x$, $D^{\beta}_{b-} x$, $I^{\beta}_{b-} x$ denote respectively: the left-sided, the right-sided Riemann-Liouville fractional derivative and the right-sided fractional integral of order β of a function x. The above problem can be viewed as a fractional counterpart of the classical Dirichlet problem of order $2n$ (for $\alpha = n$ we can identify $D^{\alpha}_{a+} x$ with classical derivative $D^n x$ and $D^{\alpha}_{b-} x$ with $(-1)^n D^n x$) of the form

$$\begin{aligned} \Big(\ldots \left(x^{(n+1)} - F_{x_1} \left(x^{(n-1)}, \ldots, x', x, t \right) \right)' + \cdots + (-1)^{n-1} F_{x_{n-1}} \left(x^{(n-1)}, \ldots, x', x, t \right) \Big)' \\ = (-1)^{n-1} F_{x_n} \left(x^{(n-1)}, \ldots, x', x, t \right) \qquad \text{for a.e. } t \in [a,b] \end{aligned}$$

$$x^{(i)}(a) = 0 \text{ and } x^{(i)}(b) = 0 \text{ for } i = 0, \ldots, n-1.$$

In the classical case, for $n = 2$ we get the following $4 - th$ order problem

$$\left(x''' - F_{x_1}\left(x', x, t\right)\right)' = -F_{x_2}\left(x', x, t\right),$$
$$x\left(a\right) = x'\left(a\right) = x\left(b\right) = x'\left(b\right) = 0.$$

In particular, for $F\left(x_1, x_2, t\right) := \frac{1}{2}q\left(t\right)x_2^2 + \varphi\left(t\right)x_2$, where q and φ are given, we get a beam equation of the form

$$x^{(4)} + q\left(t\right)x + \varphi\left(t\right) = 0$$

with boundary conditions $x\left(a\right) = x'\left(a\right) = x\left(b\right) = x'\left(b\right) = 0$.

The main result of the paper is a theorem on the existence of solutions to (1.1)–(1.2). To prove it we use a variational method relying on the showing that a functional of action possesses at least one minimum, which point generates solutions to a considered problem (see [3, 4] for the details). It should be emphasized that the above method can not guarantee that the functions $x^{(n+1)}, \ldots, x^{(2n)}$ and $F_{x_1}, \ldots F_{x_{n-1}}$ are differentiable, only the functions in brackets on the right-hand side of (1.1) are differentiable, therefore equation (1.1) is given in such a complicated form.

In the proof of the fact that the aforementioned functional of action attains its minimum we use the following

Proposition 1.1 (see [3])**.** *If X is a reflexive Banach space and the functional $\mathcal{L} : X \to \mathbb{R}$ is coercive and sequentially weakly lower semicontinuous, then it possesses at least one minimum at $x_0 \in X$.*

Let us remind that a functional $\mathcal{L}$ defined on a Banach space X is coercive if $\mathcal{L}\left(x\right) \to \infty$ whenever $\|x\| \to \infty$, and $\mathcal{L}$ is sequentially weakly lower semicontinuous at $x_0 \in X$ if $\liminf_{n\to\infty} \mathcal{L}\left(x_n\right) \geq \mathcal{L}\left(x_0\right)$ for any sequence $\{x_n\} \subset X$ such that $x_n \rightharpoonup x_0$ weakly in X.

To prove that a minimum point $\bar{x}$ gives as a solution it is sufficient to guarantee that $\mathcal{L}$ possesses the Lagrange variation at $\bar{x}$ and to apply a fractional version of du Bois-Reymond lemma proved in the paper [1].

Let $\alpha > 0$, $\varphi \in L^1([a,b], \mathbb{R}^m)$. To begin with, we shall remind some facts concerning the notions of fractional integral and derivatives. We define a left-sided Riemann-Liouville fractional integral of φ on the interval $[a,b]$ as a function $I^\alpha_{a+}\varphi$ given by

$$(I^\alpha_{a+}\varphi)(t) := \frac{1}{\Gamma(\alpha)}\int_a^t \frac{\varphi(\tau)}{(t-\tau)^{1-\alpha}}d\tau,\ t \in [a,b] \text{ a.e.}$$

and a right-sided Riemann-Liouville fractional integral of φ on the interval $[a,b]$ — $I^\alpha_{b-}\varphi$ given by

$$(I^\alpha_{b-}\varphi)(t) := \frac{1}{\Gamma(\alpha)}\int_t^b \frac{\varphi(\tau)}{(\tau-t)^{1-\alpha}}d\tau,\ t \in [a,b] \text{ a.e.}$$

We posit that $I^0_{a+}\varphi = I^0_{b-}\varphi = \varphi$.

Remark 1.2. Let us notice that if $\varphi \in L^1([a,b],\mathbb{R}^m)$ then for $\alpha = n \in \mathbb{N}$ we have

$$I^n_{a+}\varphi(t) = \frac{1}{(n-1)!}\int_a^t \frac{\varphi(s)}{(t-s)^{1-n}}ds = \int_a^t\int_a^{t_1}\dots\int_a^{t_{n-1}}\varphi(s)\,ds\,dt_{n-1}\cdots dt_1,$$

for $t \in [a,b]$. The above formula is known as the Cauchy formula. In this way the definition of $I^\alpha_{a+}\varphi$ is a natural generalization of the Cauchy formula for non-integer α (since $\Gamma(n) = (n-1)!$). In what follows we shall use the symbol I^n to denote I^α_{a+} for $\alpha = n$

For $1 \le p < \infty$ let us introduce the following notations

$$I^\alpha_{a+}(L^p) := \{I^\alpha_{a+}\varphi : \ \varphi \in L^p([a,b],\mathbb{R}^m)\},$$

$$I^\alpha_{b-}(L^p) := \{I^\alpha_{b-}\varphi : \ \varphi \in L^p([a,b],\mathbb{R}^m)\}.$$

We have the following properties of fractional integral

Proposition 1.3 (see [1] and references therein)**.** (a) *If $\alpha > 0$ and $1 \le p < \infty$, then $I^\alpha_{a+}\varphi \in L^p([a,b],\mathbb{R}^m)$ for any $\varphi \in L^p([a,b],\mathbb{R}^m)$.*

(b) *If $\alpha > 0$, $1 \le p \le \infty$ and $\alpha > \frac{1}{p}$, then for any $\varphi \in L^p([a,b],\mathbb{R}^m)$ the function $I^\alpha_{a+}\varphi$ is continuous on $[a,b]$.*

Proposition 1.4. *If $\varphi \in L^p([a,b],\mathbb{R}^m)$ with $1 \le p \le \infty$ then*

$$\left(I^\alpha_{a+}I^\beta_{a+}\varphi\right)(t) = I^{\alpha+\beta}_{a+}\varphi(t) \qquad \left(I^\alpha_{b-}I^\beta_{b-}\varphi\right)(t) = I^{\alpha+\beta}_{b-}\varphi(t)$$

for a.e. $t \in [a,b]$ and $\alpha,\beta > 0$. If moreover $\alpha+\beta > 1$ then the above equalities hold true at any $t \in [a,b]$.

Now, let us remind the definition of the fractional derivative. Let $\alpha \in (n-1,n)$, $n \in \mathbb{N}$, $x \in L^1([a,b],\mathbb{R}^m)$. We say that x possesses a left-sided Riemann-Liouville derivative $D^\alpha_{a+}x$ of order α on the interval $[a,b]$, if the function $I^{n-\alpha}_{a+}x$ is absolutely continuous on $[a,b]$, moreover

$$(D^\alpha_{a+}x)(t) := \left(D^n I^{n-\alpha}_{a+}\right)x(t) = \frac{1}{\Gamma(n-\alpha)}\frac{d^n}{dt^n}\int_a^t \frac{x(\tau)}{(t-\tau)^{1-n+\alpha}}d\tau,\ t \in [a,b] \text{ a.e.}$$

We say that x possesses a right-sided Riemann-Liouville derivative $D^\alpha_{b-}x$ of order α on the interval $[a,b]$, if the function $I^{n-\alpha}_{b-}x$ is absolutely continuous on $[a,b]$; by this derivative we mean the function $(-1)^n D^n(I^{n-\alpha}_{b-}x)$, i.e.

$$(D^\alpha_{b-}x)(t) := \frac{1}{\Gamma(n-\alpha)}(-1)^n\frac{d^n}{dt^n}\int_t^b \frac{x(\tau)}{(\tau-t)^{1-(n-\alpha)}}d\tau,\ t \in [a,b] \text{ a.e.}$$

We have the following connection between fractional derivative and fractional integral

Proposition 1.5 (see [2]). *If $\alpha > 0$ and $x \in L^p([a,b], \mathbb{R}^m)$ with $1 \le p \le \infty$ then*

$$\left(D^{\alpha}_{a+} I^{\alpha}_{a+} x\right)(t) = x(t) \text{ and } \left(D^{\alpha}_{b-} I^{\alpha}_{b-} x\right)(t) = x(t)$$

for $t \in [a,b]$ a.e. Moreover, we have that

$$\left(I^{\alpha}_{a+} D^{\alpha}_{a+} x\right)(t) = x(t) \tag{1.4}$$

for $t \in [a,b]$ a.e. provided $x \in I^{\alpha}_{a+}(L^p)$ and

$$\left(I^{\alpha}_{b-} D^{\alpha}_{b-} x\right)(t) = x(t) \tag{1.5}$$

for $t \in [a,b]$ a.e. provided $x \in I^{\alpha}_{b-}(L^p)$. What is more, if we additionally assume that $\alpha > \frac{1}{p}$, then (1.4) and (1.5) hold true for every $t \in [a,b]$.

Proposition 1.6 (see [2]). *Let $k \in \mathbb{N}$ and $\alpha \ge 0$. If the fractional derivatives $D^{\alpha}_{a+}x$ and $D^{\alpha+k}_{a+}x$ exist, then*

$$D^k D^{\alpha}_{a+} x = D^{k+\alpha}_{a+} x.$$

Finally, we have

Proposition 1.7. *Let $\alpha \in (n-1, n)$, $n = 2, 3, \ldots$, $1 \le p < \infty$, $p > \frac{1}{\alpha}$. If $x \in I^{\alpha}_{a+}(L^p)$ then:*

1. *$x \in I^{i}_{a+}(L^p)$, for $i = 1, 2, \ldots, n-1$, in particular x is absolutely continuous together with classical derivatives up to order $n-2$ and $D^{n-1}x \in L^p$.*
2. *$(D^i x)(t) = \left(I^{\alpha-i}_{a+} D^{\alpha}_{a+} x\right)(t)$ for $t \in [a,b]$ and $i = 0, \ldots, n-2$ and for a.e. $t \in [a,b]$ if $i = n-1$.*
3. *There exists $D^{\alpha-i}_{a+}x$ and $D^{\alpha-i}_{a+}x = I^i D^{\alpha}_{a+} x \in I^{i}_{a+}(L^p)$, for $i = 1, \ldots, n-1$.*

Proof.

(ad 1) If $x \in I^{\alpha}_{a+}(L^p)$ then $x(t) = \left(I^{\alpha}_{a+}\varphi\right)(t)$ for $t \in [a,b]$ (thanks to Proposition 1.3 $\left(I^{\alpha}_{a+}\varphi\right)$ is continuous, therefore x can be identified with its continuous representant) and for some $\varphi \in L^p$ (in fact $\varphi = D^{\alpha}_{a+}x$). We have that $\alpha > 1$, thus applying Proposition 1.4 we get that

$$x(t) = \left(I^{\alpha}_{a+}\varphi\right)(t) = \left(I^i I^{\alpha-i}_{a+}\varphi\right)(t) = \left(I^i \psi\right)(t),$$

for $t \in [a,b]$, where $\psi = I^{\alpha-i}_{a+}\varphi = I^{\alpha-i}_{a+}\left(D^{\alpha}_{a+}x\right) \in L^p$, for $i = 1, \ldots, n-1$.

(ad 2) It follows from the fact that

$$x(t) = \left(I^i \psi\right)(t) = \left(I^i \left(I^{\alpha-i}_{a+} D^{\alpha}_{a+} x\right)\right)(t)$$

for $t \in [a,b]$ and $i = 1, \ldots, n-1$.

(ad 3) Thanks to Proposition 1.4 we have

$$\left(I^{n-i-(\alpha-i)}x\right)(t) = \left(I^{n-\alpha}_{a+}x\right)(t) = \left(I^{n-\alpha}_{a+} I^{\alpha}_{a+}\varphi\right)(t) = \left(I^n \varphi\right)(t),$$

for $t \in [a,b]$ (where $x = I^{\alpha}_{a+}\varphi$) therefore there exists $D^{\alpha-i}_{a+}x$ for $i = 1\ldots, n-1$ and

$$\left(D^{\alpha-i}_{a+}x\right)(t) = \left(D^{n-i}I^{n-i-(\alpha-i)}_{a+}x\right)(t) = \left(D^{n-i}I^{n}\varphi\right)(t) = \left(I^{i}\varphi\right)(t) = \left(I^{i}D^{\alpha}_{a+}x\right)(t)$$

for $t \in [a,b]$. □

Next, for $\alpha \in (n-1,n)$, $n \in \mathbb{N}$, by $AC^{\alpha}_{a+}\left([a,b],\mathbb{R}^m\right)$ (simply AC^{α}_{a+}) let us denote the set of all functions $x : [a,b] \to \mathbb{R}^m$ such that there are constants $c_{n-1},\ldots,c_0 \in \mathbb{R}^m$ and a function $\varphi \in L^1([a,b],\mathbb{R}^m)$ such that

$$x(t) = \sum_{i=0}^{n-1} \frac{c_i}{\Gamma(\alpha-n+i+1)}(t-a)^{\alpha-n+i} + I^{\alpha}_{a+}\varphi,\ t \in [a,b]\ \ a.e. \tag{1.6}$$

Similarly by $AC^{\alpha}_{b-}\left([a,b],\mathbb{R}^m\right)$ (simply AC^{α}_{b-}) we denote the set of all functions $x : [a,b] \to \mathbb{R}^m$ such that there are constants $d_{n-1},\ldots,d_0 \in \mathbb{R}^m$ and a function $\psi \in L^1([a,b],\mathbb{R}^m)$ such that

$$x(t) = \sum_{i=0}^{n-1} \frac{d_i}{\Gamma(\alpha-n+i+1)}(b-t)^{\alpha-n+i} + I^{\alpha}_{b-}\varphi,\ t \in [a,b]\ \ a.e. \tag{1.7}$$

Finally, by $AC^{\alpha,p}_{a+}$ and by $AC^{\alpha,p}_{b-}$ with $1 \le p \le \infty$ we denote the set of all functions satisfying (1.6) and (1.7) respectively with a function $\varphi \in L^p([a,b],\mathbb{R}^m)$, $\psi \in L^p([a,b],\mathbb{R}^m)$ resp.

With the aid of above sets it is easy to formulate a necessary and sufficient condition for the existence of fractional derivative. Namely, we have

Theorem 1.8 (see [1])**.** *Let $\alpha \in (n-1,n)$, $n \in \mathbb{N}$, then a function x possesses a left-sided Riemann-Liouville derivative $D^{\alpha}_{a+}x$ on the interval $[a,b]$ if and only if $x \in AC^{\alpha}_{a+}\left([a,b],\mathbb{R}^m\right)$. Moreover,*

$$\left(D^{a}_{a+}x\right)(t) = \varphi(t),\ \textit{for}\ t \in [a,b]\ \ a.e.$$

and

$$\left(D^{i}I^{n-\alpha}_{a+}x\right)(a) = c_i$$

for $i = 0,\ldots,n-1$, where x is of the form (1.6). Similarly a function x possesses a right-sided Riemann-Liouville derivative $D^{\alpha}_{b-}x$ on the interval $[a,b]$ if and only if $x \in AC^{\alpha}_{b-}\left([a,b],\mathbb{R}^m\right)$. Moreover,

$$\left(D^{a}_{b-}x\right)(t) = \psi(t),\ \textit{for}\ t \in [a,b]\ \ a.e.$$

and

$$\left(D^{i}I^{n-\alpha}_{b-}x\right)(b) = (-1)^{i}d_i$$

for $i = 0,\ldots,n-1$, where x is of the form (1.7).

The sets $AC^{\alpha,p}_{a+}([a,b],\mathbb{R}^m)$ and $AC^{\alpha,p}_{b-}([a,b],\mathbb{R}^m)$ are linear spaces, moreover these spaces equipped with the norms

$$\|x\|_+ = \sum_{i=0}^{n-1} \left|\left(D^i I^{n-\alpha}_{a+} x\right)(a)\right| + \left\|D^a_{a+} x\right\|_{L^p}, \tag{1.8}$$

$$\|x\|_- = \sum_{i=0}^{n-1} \left|\left(D^i I^{n-\alpha}_{b-} x\right)(b)\right| + \left\|D^a_{b-} x\right\|_{L^p} \tag{1.9}$$

are Banach spaces.

Note that the spaces $AC^{\alpha,2}_{a+}([a,b],\mathbb{R}^m)$ and $AC^{\alpha,2}_{b-}([a,b],\mathbb{R}^m)$ equipped with the following inner products

$$\langle x,y\rangle_+ = \sum_{i=0}^{n-1} \left\langle \left(D^i I^{n-\alpha}_{a+} x\right)(a), \left(D^i I^{n-\alpha}_{a+} y\right)(a)\right\rangle + \int_a^b \left\langle D^a_{a+} x(t)\, D^a_{a+} y(t)\right\rangle dt,$$

$$\langle x,y\rangle_- = \sum_{i=0}^{n-1} \left\langle \left(D^i I^{n-\alpha}_{b-} x\right)(b), \left(D^i I^{n-\alpha}_{b-} y\right)(b)\right\rangle + \int_a^b \left\langle D^a_{b-} x(t)\, D^a_{b-} y(t)\right\rangle dt$$

are Hilbert spaces (the norms defined by the above inner products are equivalent to the norms given by (1.8) and (1.9)).

Moreover, we have the following simple characterization of the weak convergence in $AC^{\alpha,2}_{a+}$ (and analogously, in the space $AC^{\alpha,2}_{b-}$).

Proposition 1.9. *A sequence $\{x_k\} \subset AC^{\alpha,2}_{a+}$ tends to $x_0 \in AC^{\alpha,2}_{a+}$ weakly in $AC^{\alpha,2}_{a+}$ (we shall write $x_k \rightharpoonup x_0$ to denote it) if and only if*

$$\left\langle \left(D^i I^{n-\alpha}_{a+} x_n\right)(a), c\right\rangle_{\mathbb{R}^m} \to \left\langle \left(D^i I^{n-\alpha}_{a+} x_0\right)(a), c\right\rangle_{\mathbb{R}^m}$$

and $\int_a^b D^a_{a+} x_n(t)\varphi(t)\,dt \to \int_a^b D^a_{a+} x_0(t)\varphi(t)\,dt$ for any $c \in \mathbb{R}^m$, $i = 0,1,\dots,n-1$ and any $\varphi \in L^2$.

2. THE SPACE OF SOLUTIONS

Let $\alpha \in (n-1,n)$, with $n = 2,\dots$. Let us consider the following boundary problem

(2.1)
$$\begin{aligned} D^1\Big(\dots D^1\Big(D^{\alpha-(n-1)}_{b-} D^\alpha_{a+} x(t) - I^{n-\alpha}_{b-} F_{x_1}\left(D^{\alpha-1}_{a+} x(t),\dots,D^{\alpha-(n-1)}_{a+} x(t), x(t), t\right)\Big) \\ + \dots + (-1)^{n-1} I^{n-\alpha}_{b-} F_{x_{n-1}}\left(D^{\alpha-1}_{a+} x(t),\dots,D^{\alpha-(n-1)}_{a+} x(t), x(t), t\right)\Big) \\ = (-1)^{n-1} F_{x_n}\left(D^{\alpha-1}_{a+} x(t),\dots,D^{\alpha-(n-1)}_{a+} x(t), x(t), t\right) \qquad \text{for a.e. } t \in [a,b] \end{aligned}$$

$$\left(D^i I^{n-\alpha}_{a+} x\right)(a) = 0 \text{ and } \left(I^{a-i} D^\alpha_{a+} x\right)(b) = 0 \text{ for } i = 0,\dots,n-1 \tag{2.2}$$

where $F : (\mathbb{R}^m)^n \times [a,b] \to \mathbb{R}^m$.

We seek solutions to (2.1)–(2.2) in the space

$$H_0^\alpha([a,b],\mathbb{R}^m) = H_0^\alpha$$
$$= \left\{x \in AC_{a+}^{\alpha,2} : \left(D^i I_{a+}^{n-\alpha} x\right)(a) = 0 \text{ and } \left(I^{a-i} D_{a+}^\alpha x\right)(b) = 0 \text{ for } i = 0, \ldots, n-1\right\}.$$

Remark 2.1. If $x \in H_0^\alpha$ then $x \in I_{a+}^\alpha(L^2)$, consequently for $x \in H_0^\alpha$ all conclusions of Proposition 1.3 are true.

We have the following

Lemma 2.2 (Fractional Poincaré Inequality)**.** *If* $\alpha \in \left(n - \frac{1}{2}, n\right)$, $n \in \mathbb{N}$, *then*

$$\left(\int_a^b |x(t)|^2 dt\right)^{\frac{1}{2}} \leq C_\alpha \left(\int_a^b \left|\left(D_{a+}^\alpha x\right)(t)\right|^2 dt\right)^{\frac{1}{2}}$$

for $x \in I_{a+}^\alpha(L^2)$, *where* $C_\alpha := \frac{(b-a)^\alpha}{\Gamma(\alpha)\sqrt{2\alpha-1}}$.

Proof. First, let us notice that since $\alpha > \frac{1}{2}$ therefore the function $(a,t) \ni s \mapsto (t-s)^{\alpha-1}$ belongs to the space $L^2([a,t],\mathbb{R})$. The application of Proposition 1.5 and the Hölder inequality lead to

$$\begin{aligned} |x(t)| \leq \left|\left(I_{a+}^\alpha D_{a+}^\alpha x\right)(t)\right| &= \frac{1}{\Gamma(\alpha)} \int_a^t \frac{\left|\left(D_{a+}^\alpha x\right)(s)\right|}{(t-s)^{1-\alpha}} ds \\ &\leq \frac{1}{\Gamma(\alpha)} \left(\int_a^t (t-s)^{2(\alpha-1)} ds\right)^{\frac{1}{2}} \left(\int_a^t \left|\left(D_{a+}^\alpha x\right)(s)\right|^2 ds\right)^{\frac{1}{2}} \\ &\leq \frac{(b-a)^{\alpha-\frac{1}{2}}}{\Gamma(\alpha)\sqrt{2\alpha-1}} \left(\int_a^b \left|\left(D_{a+}^\alpha x\right)(s)\right|^2 ds\right)^{\frac{1}{2}}. \end{aligned}$$

Consequently

$$\begin{aligned} \left(\int_a^b |x(t)|^2 dt\right)^{\frac{1}{2}} &\leq \sqrt{(b-a) \operatorname*{esssup}_{t\in[a,b]} |x(t)|^2} \\ &\leq \frac{(b-a)^\alpha}{\Gamma(\alpha)\sqrt{2\alpha-1}} \left(\int_a^b \left|\left(D_{a+}^\alpha x\right)(s)\right|^2 dt\right)^{\frac{1}{2}}. \end{aligned}$$

□

Remark 2.3. Let us note that if $x \in I^k(L^2)$ with $k \in \mathbb{N}$ then using the similar argumentation as in the proof of Fractional Poincaré Inequality it can be proved that

$$\left(\int_a^b |x(t)|^2 dt\right)^{\frac{1}{2}} \leq C_k \left(\int_a^b \left|x^{(k)}(t)\right|^2 dt\right)^{\frac{1}{2}}, \tag{2.3}$$

where $C_k = \frac{(b-a)^k}{(k-1)!\sqrt{2k-1}}$

As a consequence of Remark 2.3 and Propositions 1.6 and 1.7 we have the following

Lemma 2.4. *If* $x \in I^{\alpha}_{a+}(L^2)$, *where* $\alpha \in (n-1, n)$, $n \in \mathbb{N}$, *then*

$$\left(\int_a^b \left|D^{\alpha-i}_{a+}x(t)\right|^2 dt\right)^{\frac{1}{2}} \leq C_i \left(\int_a^b \left|D^{\alpha}_{a+}x(t)\right|^2 dt\right)^{\frac{1}{2}}$$

for $i = 1, \ldots, n-1$.

Remark 2.5. The constants C_i and C_α may not be optimal. For instance in the classical Poincaré Inequality for the space H^1_0 and the interval $[0, \pi]$ its is proved that the optimal constant equals 1 (see [3]). It should be emphasize that the constants are given *explicite.*

3. THE MAIN RESULT

In this section we present a general assumptions and then formulate and prove the main result of the paper.

In what follows we shall assume that:

(A1) The function F is a Carathodory function i.e. the function

$$[a, b] \ni t \mapsto F(x_1, \ldots, x_n, t)$$

is measurable for $(x_1, \ldots, x_n) \in (\mathbb{R}^m)^n$ and the function

$$(\mathbb{R}^m)^n \ni (x_1, \ldots, x_n) \mapsto F(x_1, \ldots, x_n, t)$$

is continuous for a.e. $t \in [a, b]$.

(A2)

$$F(x_1, \ldots, x_n, t) \leq \sum_{i=1}^{n} \frac{a_i}{2} |x_i|^2 + \sum_{i=1}^{n} b_i |x_i| + \gamma(t),$$

for $x_i \in \mathbb{R}^m$, a.e. $t \in [a, b]$, where $a_i > 0$, $b_i \in \mathbb{R}$, for $i = 0, 1, \ldots, n-1$ and $\gamma \in L^1([a, b], \mathbb{R})$.

(A3) The coefficients $a_i > 0$, $b_i \in \mathbb{R}$, $i = 0, 1, \ldots, n-1$ satisfy the following relation

$$1 - a_n C^2_\alpha - \sum_{i=1}^{n-1} a_i C^2_i > 0,$$

where C_i, $i = 0, 1, \ldots, n-1$ and C_α are given in the Poincaré Inequalities (see Remark 2.5).

(A4) The function F possesses partial derivatives F_{x_i}, for a.e. $t \in [a, b]$ and every $x_i \in \mathbb{R}^m$, $i = 0, 1, \ldots, n-1$ and every F_{x_i} is a Carathodory function.

(A5) For any $r > 0$ there is a function $\gamma_r \in L^1([a, b], \mathbb{R})$ such that

$$|F(x_1, \ldots, x_n, t)| \leq \gamma_r(t)$$
$$|F_{x_i}(x_1, \ldots, x_n, t)| \leq \gamma_r(t)$$

for a.e. $t \in [a, b]$ and every $x_i \in \mathbb{R}^m$ such that $|x_i| \leq r$, $i = 0, 1, \ldots, n-1$.

The proof of the main theorem on the existence of solution to (2.1)–(2.2) is based on some variational methods. Namely we define a functional $\mathcal{L}: H_0^\alpha \to \mathbb{R}$ of the form

$$\mathcal{L}(x) = \frac{1}{2}\int_a^b \left|D_{a+}^\alpha x(t)\right|^2 dt - \int_a^b F\left(D_{a+}^{\alpha-1}x(t), \dots, D_{a+}^{\alpha-(n-1)}x(t), x(t), t\right) dt$$

and then we prove that $\mathcal{L}$ possesses a minimum point $x \in H_0^\alpha$. Finally, application of a fractional version of the fundamental lemma gives us a solution. For the convenience of the reader we formulate the aforementioned lemma

Lemma 3.1 (Fundamental Lemma (see [1])). *If $\alpha \in \left(n-\frac{1}{2}, n\right) < n$, $n = 2, \dots$, $\alpha_0, \alpha_1, \beta_1, \dots, \beta_{n-1} \in L^2([a,b], \mathbb{R}^m)$ and*

$$\int_a^b (\alpha_1(t)(D_{a+}^\alpha h)(t)dt - \sum_{i=1}^{n-1}\int_a^b \beta_i(t)(D_{a+}^{\alpha-i}h)(t)dt - \int_a^b \alpha_0(t)h(t)dt = 0$$

for any $h \in H_0^\alpha$, then there exists the derivative $D_{b-}^{\alpha-(n-1)}\alpha_1$, the functions

$$D_{b-}^{\alpha-(n-1)}\alpha_1 - I_{b-}^{n-\alpha}\beta_1$$
$$D^1(D_{b-}^{\alpha-(n-1)}\alpha_1 - I_{b-}^{n-\alpha}\beta_1) + I_{b-}^{n-\alpha}\beta_2$$
$$D^1(D^1(D_{b-}^{\alpha-(n-1)}\alpha_1 - I_{b-}^{n-\alpha}\beta_1) + I_{b-}^{n-\alpha}\beta_2) - I_{b-}^{n-\alpha}\beta_3$$
$$\vdots$$
$$D^1(\dots D^1(D_{b-}^{\alpha-(n-1)}\alpha_1 - I_{b-}^{n-\alpha}\beta_1) + \dots + (-1)^{n-2}I_{b-}^{n-\alpha}\beta_{n-2}) + (-1)^{n-1}I_{b-}^{n-\alpha}\beta_{n-1}$$

are absolutely continuous and

$$D^1\left(\dots D^1\left(D_{b-}^{\alpha-(n-1)}\alpha_1 - I_{b-}^{n-\alpha}\beta_1\right) + \dots + (-1)^{n-1}I_{b-}^{n-\alpha}\beta_{n-1})\right) = (-1)^{n-1}a_0$$

a.e. on $[a,b]$ (the operator D^1 acts $(n-1)$ times).

In the proof of the main theorem we use the following consequences of the weak convergence in the space $I_{a+}^\alpha(L^2)$.

Lemma 3.2. *Suppose that $\alpha \in \left(n-\frac{1}{2}, n\right)$, $n = 2, \dots$, $p > 1$. If $\{x_k\} \subset I_{a+}^\alpha(L^2)$ and $x_k \rightharpoonup x_0 \in I_{a+}^\alpha(L^2)$ weakly in $I_{a+}^\alpha(L^2)$, then $x_k \rightrightarrows x_0$ and $D_{a+}^{\alpha-i}x_k \rightrightarrows D_{a+}^{\alpha-i}x_0$ uniformly on $[a,b]$ for $i = 1, \dots, n-1$.*

Proof. Fix $i \in \{1, \dots, n-1\}$. To begin with we will prove that $\{x_k\}$ and $\left\{D_{a+}^{\alpha-i}x_k\right\}$ are relatively compact in the topology of the space $C([a,b], \mathbb{R}^m)$ of continuous function with the standard supremum norm (note that in virtue of Proposition 1.7 x, $D_{a+}^{\alpha-i}x \in I_{a+}^i(L^2) \subset C([a,b], \mathbb{R}^m)$). Since $\{x_k\}$ is weakly convergent, therefore it is bounded, consequently taking into account the same argumentation as in the proof of Lemma 2.2 we get

$$|x_k(t)| \le C\|x_k\|_+ = c,$$

for $t \in [a,b]$ and $k \in \mathbb{N}$, where $C > 0$. Next, thanks to Proposition 1.7 (3) and the Hölder inequality, we get for $a \leq t_1 \leq t_2 \leq b$ and $k \in \mathbb{N}$ that

$$\begin{aligned}
&\left|\left(D_{a+}^{\alpha-i} x_k\right)(t_1) - \left(D_{a+}^{\alpha-i} x_k\right)(t_2)\right| = \left|\left(I^i D_{a+}^{\alpha} x_k\right)(t_1) - \left(I^i D_{a+}^{\alpha} x_k\right)(t_2)\right| \\
&= \frac{1}{(i-1)!}\left|\int_a^{t_1} \frac{\left(D_{a+}^{\alpha} x_k\right)(s)}{(t_1-s)^{1-i}} ds - \int_a^{t_2} \frac{\left(D_{a+}^{\alpha} x_k\right)(s)}{(t_2-s)^{1-i}} ds\right| \\
&\leq \frac{1}{(i-1)!}\int_a^{t_1} \left|\left(D_{a+}^{\alpha} x_k\right)(s)\right| \left|(t_1-s)^{i-1} - (t_2-s)^{i-1}\right| ds \\
&\quad + \frac{1}{(i-1)!}\int_{t_1}^{t_2} \frac{\left|\left(D_{a+}^{\alpha} x_k\right)(s)\right|}{(t_2-s)^{1-i}} ds \\
&\leq \frac{1}{(i-1)!}\left(\int_a^{t_1} \left|\left(D_{a+}^{\alpha} x_k\right)(s)\right|^2 ds\right)^{\frac{1}{2}} \left(\int_a^{t_1} \left|(t_2-s)^{2i-2} - (t_1-s)^{2i-2}\right| ds\right)^{\frac{1}{2}} \\
&\quad + \frac{1}{(i-1)!}\left(\int_{t_1}^{t_2} \left|\left(D_{a+}^{\alpha} x_k\right)(s)\right|^2 ds\right)^{\frac{1}{2}} \left(\int_{t_1}^{t_2} (t_2-s)^{2i-2} ds\right)^{\frac{1}{2}} \\
&\leq \|x_k\|_+ \frac{1}{(i-1)!\sqrt{2i-1}}\left(\left((t_2-a)^{2i-1} - (t_1-a)^{2i-1} - (t_2-t_1)^{2i-1}\right)\right)^{\frac{1}{2}} \\
&\quad + \|x_k\|_+ \frac{1}{(i-1)!\sqrt{2i-1}} (t_2-t_1)^{i-\frac{1}{2}} \\
&\leq \bar{c}\sqrt{\left|(t_1-a)^{2i-1} - (t_2-a)^{2i-1}\right|} + 2\bar{c}\,(t_2-t_1)^{i-\frac{1}{2}}.
\end{aligned}$$

Consequently, the sequence $\left\{D_{a+}^{\alpha-i} x_k\right\}$ is equicontinuous. In a similar fashion it can be proved that the sequence $\left\{D_{a+}^{\alpha-i} x_k\right\}$ is equibounded. Using Arzelà-Ascoli theorem we get that $\left\{D_{a+}^{\alpha-i} x_k\right\}$ is relatively compact. In the same way, applying equality $x_k = I_{a+}^{\alpha} D_{a+}^{a} x_k$, we get that $\{x_k\}$ is also relatively compact. It is easy to notice that $x_k \rightharpoonup x_0$ weakly in $C([a,b], \mathbb{R}^m)$ therefore each subsequence of $\{x_k\}$ must be convergent in $C([a,b], \mathbb{R}^m)$ to x_0 which means that $x_k \to x_0$ in $C([a,b], \mathbb{R}^m)$.

Moreover, since

$$\sup_{t\in[a,b]} \left|D_{a+}^{\alpha-i} x(t)\right| \leq c\,\|x\|_+$$

for $x \in I_{a+}^{\alpha}(L^2)$ therefore the operator $T : I_{a+}^{\alpha}(L^2) \ni x \mapsto D_{a+}^{\alpha-i} x(t) \in C([a,b], \mathbb{R}^m)$ is linear and continuous and, as a consequence, $D_{a+}^{\alpha-i} x_k \rightharpoonup D_{a+}^{\alpha-i} x_0$ weakly in $C([a,b], \mathbb{R}^m)$. Thus, $D_{a+}^{\alpha-i} x_k \to D_{a+}^{\alpha-i} x_0$ in $C([a,b], \mathbb{R}^m)$. □

Now, we can prove the main result of the paper.

Theorem 3.3. *Assume* (A1)–(A5) *then problem* (2.1)–(2.2) *possesses at least one solution which minimizes functional* $\mathcal{L}$.

Proof. Let $x \in H_0^\alpha$, then by (A2) and applying Hölder and Poincaré inequalities we get

$$\begin{aligned}
\mathcal{L}(x) &= \frac{1}{2}\int_a^b \left|D_{a+}^{\alpha}x(t)\right|^2 dt - \int_a^b F\left(D_{a+}^{\alpha-1}x(t),\ldots,D_{a+}^{\alpha-(n-1)}x(t),x(t),t\right)dt \\
&\geq \frac{1}{2}\|x\|_+^2 - \frac{a_n}{2}\int_a^b |x(t)|^2 dt - \sum_{i=1}^{n-1}\frac{a_i}{2}\int_a^b \left|D_{a+}^{\alpha-i}x(t)\right|^2 dt \\
&\quad - b_n\int_a^b |x(t)|\,dt - \sum_{i=1}^{n-1} b_i\int_a^b \left|D_{a+}^{\alpha-i}x(t)\right| dt - \int_a^b \gamma(t)\,dt \\
&\geq \frac{1}{2}\|x\|_+^2 - \frac{a_n}{2}C_\alpha^2\int_a^b \left|D_{a+}^{\alpha}x(t)\right|^2 dt - \sum_{i=1}^{n-1}\frac{a_i}{2}C_i^2\int_a^b \left|D_{a+}^{\alpha}x(t)\right|^2 dt \\
&\quad - b_n\sqrt{b-a}\left(\int_a^b |x(t)|^2 dt\right)^{\frac{1}{2}} - \sum_{i=1}^{n-1} b_i\sqrt{b-a}\left(\int_a^b \left|D_{a+}^{\alpha-i}x(t)\right|^2 dt\right)^{\frac{1}{2}} - \|\gamma\|_{L^1} \\
&\geq \frac{1}{2}\left(1 - a_nC_\alpha^2 - \sum_{i=1}^{n-1}a_iC_i^2\right)\|x\|_+^2 - \sqrt{(b-a)}\left(C_\alpha b_n + \sum_{i=1}^{n-1}C_ib_i\right)\|x\|_+ - \|\gamma\|_{L^2}.
\end{aligned}$$

Consequently, by (A3) $\mathcal{L}$ is coercive. Let $x_k \rightharpoonup x_0$ weakly in H_0^α. From Lemma 3.2 it follows that $x_k \rightrightarrows x$ and $D_{a+}^{\alpha-i}x_k \rightrightarrows D_{a+}^{\alpha-i}x_0$ thus by the dominated convergence theorem (we use here assumption (A5)) we get that

$$\begin{aligned}
&\lim_{k\to\infty}\int_a^b F\left(D_{a+}^{\alpha-1}x_k(t),\ldots,D_{a+}^{\alpha-(n-1)}x_k(t),x_k(t),t\right)dt \\
&\qquad\qquad = \int_a^b F\left(D_{a+}^{\alpha-1}x_0(t),\ldots,D_{a+}^{\alpha-(n-1)}x_0(t),x_0(t),t\right)dt.
\end{aligned}$$

Moreover, since the function $x \mapsto \|x\|_+$ is weakly lower semicontinuous therefore we have

$$\begin{aligned}
\liminf_{k\to\infty} J(x_k) &= \liminf_{k\to\infty}\frac{1}{2}\|x_k\|_+^2 + \lim_{k\to\infty}\int_a^b F\left(D_{a+}^{\alpha-1}x_k(t),\ldots,D_{a+}^{\alpha-(n-1)}x_k(t),x_k(t),t\right) \\
&\geq \frac{1}{2}\|x_0\|_+^2 + \int_a^b F\left(D_{a+}^{\alpha-1}x_0(t),\ldots,D_{a+}^{\alpha-(n-1)}x_0(t),x_0(t),t\right)dt = J(x_0),
\end{aligned}$$

thus the functional $\mathcal{L}$ is sequentially weakly lower semicontinuous and in virtue of Proposition 1.1 we have that it possesses minimum at a point $\bar{x} \in H_0^\alpha$.

Thanks to (A4) and (A5) we have that the Lagrange variation $\Phi_{\mathcal{L}}(x,h)$ of $\mathcal{L}$ exists at any point $x \in H_0^\alpha$ and at any direction $h \in H_0^\alpha$, thus using Fermat lemma we get that

$$\begin{aligned}
&\int_a^b \left(D_{a+}^{\alpha}\bar{x}\right)(t)\left(D_{a+}^{\alpha}h\right)(t)\,dt \\
&\quad - \sum_{i=1}^{n-1}\int_a^b F_{x_i}\left(D_{a+}^{\alpha-1}\bar{x}(t),\ldots,D_{a+}^{\alpha-(n-1)}\bar{x}(t),\bar{x}(t),t\right)D_{a+}^{\alpha-i}h(t)\,dt
\end{aligned}$$

$$= \int_a^b F_{x_n}\left(D_{a+}^{\alpha-1}\bar{x}(t),\ldots,D_{a+}^{\alpha-(n-1)}\bar{x}(t),\bar{x}(t),t\right)h(t)\,dt$$

for $h \in H_0^\alpha$. Applying fundamental lemma 3.1 we obtain that there exists $D_{b-}^{\alpha-(n-1)}D_{a+}^{\alpha}\bar{x}$ and

$$\begin{aligned} D^1\Big(\cdots D^1\Big(D_{b-}^{\alpha-(n-1)}D_{a+}^{\alpha}\bar{x} - I_{b-}^{n-\alpha}F_{x_1}\left(D_{a+}^{\alpha-1}\bar{x}(t),\ldots,D_{a+}^{\alpha-(n-1)}\bar{x}(t),\bar{x}(t),t\right)\Big)\\ \cdots + (-1)^{n-1}I_{b-}^{n-\alpha}F_{x_{n-1}}\left(D_{a+}^{\alpha-1}\bar{x}(t),\ldots,D_{a+}^{\alpha-(n-1)}\bar{x}(t),\bar{x}(t),t\right)\Big)\\ = (-1)^{n-1}F_{x_n}\left(D_{a+}^{\alpha-1}\bar{x}(t),\ldots,D_{a+}^{\alpha-(n-1)}\bar{x}(t),\bar{x}(t),t\right) \end{aligned}$$

The boundary conditions (2.2) are satisfied thanks to the fact that $\bar{x} \in H_0^\alpha$. □

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