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RAZUMIKHIN TECHNIQUE VIA TWO LYAPUNOV FUNCTIONS AND APPLICATIONS TO LOTKA-VOLTERRA SYSTEMS WITH TIME DELAY AND IMPULSIVE EFFECTS

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ABSTRACT. This paper considers the stability problem of a class of impulsive functional differential equations using two Lyapunov functions. By employing the Lyapunov-Razumikhin technique, we establish several stability criteria in terms of two measures. These criteria are then applied to get sufficient conditions for uniform asymptotical stability of Lotka-Volterra systems subject to impulsive effects.

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1. INTRODUCTION

The theory of impulsive functional differential equations has been developed rapidly in recent years. Existence and uniqueness results are established in [1, 9]. A series of stability results are developed in [10, 13, 15]. The study of impulsive functional differential equations is much more difficult than that of impulsive differential equations [5, 9]. This is mainly due to the discontinuities of the solutions which renders the classical techniques used in the theory of functional differential equations ineffective(see [3]).

The Lyapunov-Razumikhin technique is a powerful tool for the investigation of qualitative properties of functional differential equations and has been extended recently to the study of stability in terms of two measures([11, 12]). However, in most cases, only one Lyapunov function is used, which makes it difficult to construct appropriate Lyapunov functions, especially for real world systems. In this paper, we utilize two Lyapunov functions to investigate the stability in terms of two measures for impulsive functional differential equations based on the ideas developed in [7, 11, 12], and then apply our results to Lotka-Volterra systems subject to impulsive effects.

The rest of the paper is organized as follows. In Section 2, we introduce some notations and definitions. In Section 3, we obtain several Razumikhin-type stability criteria for impulsive functional differential equations and in Section 4, these results are applied to obtain stability properties of Lotka-Volterra systems subject to impulsive effects.

2. PRELIMINARIES

Let R denote the set of real numbers, R_+ the set of nonnegative real numbers and R^n the n-dimensional real space equipped with the Euclidean norm $||\cdot||$. Let N denote the set of positive integers, i.e., $N = \{1, 2, \cdots\}$.

We define the following classes of functions for later use.

$$\Gamma = \left\{ h \in C(R_+ \times R^n, R_+) \middle| \inf_{(t,x)} h(t,x) = 0 \right\},$$

$$\Gamma_0 = \left\{ h_0 : R_+ \times PC([-r,0], R^n) \to R_+ \middle| h_0(t,\phi) = \sup_{-r\le s\le 0} h^0(t+s, \phi(s)), \text{ where } h^0 \in \Gamma \right\},$$

$$K_0 = \left\{ g \in C(R_+, R_+) \middle| g(0) = 0 \text{ and } g(s) > 0 \text{ for } s > 0 \right\},$$

$$K = \left\{ g \in K_0 \middle| g \text{ is strictly increasing in } s \right\},$$

$$PC([a,b], S) = \left\{ \psi : [a,b] \to S \middle| \psi(t) = \psi(t^+), \forall t \in [a,b); \psi(t^-) \text{ exists in } S, \forall t \in (a,b]; \text{ and } \psi(t^-) = \psi(t) \text{ for all but at most a finite number of points } t \in (a,b] \right\},$$

$$PC([a,b), S) = \left\{ \psi : [a,b) \to S \middle| \psi(t) = \psi(t^+), \forall t \in [a,b); \psi(t^-) \text{ exists in } S, \forall t \in (a,b); \text{ and } \psi(t^-) = \psi(t) \text{ for all but at most a finite number of points } t \in (a,b) \right\},$$

and $$PC([a,\infty), S) = \left\{ \psi : [a,\infty) \to S \middle| \forall c > a, \psi|_{[a,c]} \in PC([a,c], S) \right\},$$

where $\psi(t^+) = \lim_{s\to t^+} \psi(s)$, $\psi(t^-) = \lim_{s\to t^-} \psi(s)$, $a, b \in R$ with $a < b$ and $S \subset R^n$.

Given a constant $r > 0$, we equip the linear space $PC([-r,0], R^n)$ with the norm $\|\cdot\|_r$ defined by $\|\psi\|_r = \sup_{-r\le s\le 0} \|\psi(s)\|$; when $r = \infty$, we mean $[-r,0] = (-r,0]$.

Consider the following impulsive functional differential equations

$$(2.1)\qquad \begin{cases} x'(t) &= f(t, x_t), \qquad t \in [t_{k-1}, t_k), \\ x(t_k) &= J_k(x_{t_k^-}), \qquad k \in N, \\ x_{t_0} &= \phi, \end{cases}$$

where $f : R_+ \times PC([-r, 0], R^n) \to R^n; J_k \in C(R^n, R^n), \ \phi \in PC([-r, 0], R^n)$; $0 \le t_0 < t_1 < t_2 < \cdots < t_k < \cdots$, with $t_k \to \infty$ as $k \to \infty$; $\triangle x(t) = x(t^+) - x(t^-)$; and $x_t, x_{t^-} \in PC([-r, 0], R^n)$ are defined by $x_t(s) = x(t + s), x_{t^-}(s) = x(t^- + s)$ for $-r \le s \le 0$, respectively.

In this paper, we assume that the functions $f, J_k, k \in N$ satisfy all necessary conditions for the global existence and uniqueness of solutions for all $t \ge t_0$(see [7, 10]). Denote the solution of system (2.1) by $x(t) = x(t, t_0, \phi)$ such that $x_{t_0} = \phi$. The solution $x(t)$ is continuous for $t \ne t_k, \ k \in N$, and has discontinuities of the first kind at $t = t_k$ where it is assumed to be continuous from the right, i.e., $x(t_k^+) = x(t_k), k \in N$([7, 10]).

Definition 2.1. *Let $h \in \Gamma, \ h_0 \in \Gamma_0$. Then system (2.1) is said to be*

(S1). *(h_0, h)-equi-stable(equi-S for short), if for each $\varepsilon > 0$ and $t_0 \ge 0$, there exists some $\delta = \delta(\varepsilon, t_0) > 0$, such that $h_0(t_0, \phi) < \delta$ implies $h(t, x(t)) < \varepsilon$ for $t \ge t_0$, where $x(t) = x(t, t_0, \phi)$ is any solution of system (2.1);*

(S2). *(h_0, h)-uniformly stable(US for short), if the δ in $(S1)$ is independent of t_0;*

(S3). *(h_0, h)-equi-asymptotically stable(equi-AS for short), if $(S1)$ holds and for each $\varepsilon > 0$ and $t_0 \ge 0$, there exists some $\delta = \delta(t_0) > 0$ and $T = T(\varepsilon, t_0) > 0$, such that $h_0(t_0, \phi) < \delta$ implies $h(t, x(t)) < \varepsilon$ for $t \ge t_0 + T$;*

(S4). *(h_0, h)-uniformly asymptotically stable(UAS for short), if $(S2)$ holds and for each $\gamma > 0$ and $t_0 \ge 0$, there exists some $\eta = \eta(\gamma) > 0$ and $T = T(\gamma) > 0$ such that $h_0(t_0, \phi) < \eta$ implies $h(t, x(t)) < \gamma$ for any $t \ge t_0 + T$ and $t_0 \ge 0$.*

Definition 2.2. *A function $V(t, x) : \ [t_0 - r, \infty) \times R^n \to R_+$ is said to belong to class ν_0 if*

(H1). *V is continuous on each of the sets $[t_{k-1}, t_k) \times R^n$, and for all $x, y \in R^n$ and $k \in N$, $\lim_{(t,y)\to(t_k^-,x)} V(t, y) = V(t_k^-, x)$ exists;*

(H2). *$V(t, x)$ is locally Lipschitz in $x \in R^n$.*

Definition 2.3. *Given a function $V : \ [t_0 - r, \infty) \times R^n \to R_+$, the upper right-hand derivative of V with respect to system (2.1) is defined by*

$$V'(t, \psi(0)) = \limsup_{h\to 0^+} \frac{1}{h}[V(t + h, \psi(0) + hf(t, \psi)) - V(t, \psi(0))],$$

for $(t, \psi) \in R_+ \times PC([-r, 0], R^n)$.

3. STABILITY CRITERIA

We shall state and prove our main results in this section.

Theorem 3.1. *Suppose (2.1) is* (h_0, h)*-US,* $w_i \in K,\ V,\ H \in \nu_0$ *such that*

(i) $0 \le V(t,x) \le w_1(h(t,x))$ *and*
$w_2(h(t,x)) \le H(t,x) \le w_3(h(t,x)),$ *if* $h(t,x) < \rho;$

(ii) $V(t_k, J_k(x)) \le \psi_k(V(t_k^-, x))$ *and* $H(t_k, J_k(x)) \le \psi_k(H(t_k^-, x)),$ *if* $h(t,x) < \rho,$ *where* $\psi_k \in C(R_+, R_+),\ \psi_k(s) \ge s$ *and* $\frac{\psi_k(s)}{s}$ *is nondecreasing for* $s > 0$*, and for any* $a_1 > 0$*, there is a constant* M *so that*

$$\sum_{k=1}^{\infty} [\frac{\psi_k(a_1)}{a_1} - 1] \ = \ M \ < \ \infty;$$

(iii) *there exist constants* $r,\ T^* > 0$ *and* $g \in C(R, R_+)$ *such that for any solution* $x(t)$ *of (2.1) and* $t \ge T^*$

$$\begin{aligned} V'(t,x(t)) &\le -F(t, h(t,x(t))) + g(t), \\ H'(t,x(t)) &\le -F_1(t, h(t,x(t))), \end{aligned}$$

whenever $h(t,x(t)) < \rho$ *and* $P(H(t,x(t))) > H(t+s, x(t+s))$ *for* $-r \le s \le 0,$ *where* $P \in C(R_+, R_+),\ P(s) > s$ *for* $s > 0$ *and* $F(t, h(t,x(t))) \ge \psi(t,\eta) \ge 0$ *for* $h(t,x(t)) \ge \eta > 0,$ *where* $\psi(t,\eta)$ *is measurable;* $F_1(t, h(t,x(t))) \ge 0;$

(iv) *for any given* $\eta > 0,$ $\lim_{p\to\infty} \inf_{t\ge 0} \int_t^{t+p} \psi(s,\eta)ds = \infty$ *and* $\int_0^\infty g(t)dt = \Omega < \infty;$

(v) *there exists some* $0 < \rho_0 < \rho$ *such that* $h(t_k, x) < \rho_0$ *implies* $h(t_k, J_k(x)) < \rho.$

Then (2.1) is (h_0, h)*-UAS.*

Proof. Since (2.1) is (h_0, h)-US, then for $\rho_0 > 0$, there exists a $\delta > 0$ independent of t_0, such that $h_0(t_0, \phi) < \delta$ implies $h(t, x(t)) < \rho_0$ for all $t \ge t_0$. Choose a $\beta > 0$ so that $w_3(\beta) = w_2(\rho_0)$, then if $h_0(t_0, \phi) < \delta$, we have

$$H(t, x(t)) \le w_3(\beta) \ \text{and} \ V(t, x(t)) \le w_1(\rho_0), \quad \forall t \ge t_0. \tag{3.1}$$

Thus, for any $t \ge t_0$, we have $h(t, x(t)) \le \rho_0 < \rho$.

For any $\varepsilon \in (0, \min\{\rho_0, \beta\})$, choose

$$0 < 2a < \min\Big\{w_2(\varepsilon), \quad \inf_{\frac{w_2(\varepsilon)}{2} \le s \le w_3(\beta)} \{P(s) - s\}\Big\}. \tag{3.2}$$

Since $\sum_{k=1}^{\infty} [\frac{\psi_k(a_1)}{a_1} - 1] \ < \ \infty$, there exists $K^* \in N$ such that

$$\sum_{k=K^*}^{\infty} [\frac{\psi_k(a_1)}{a_1} - 1] \ < \ \frac{a}{w_3(\beta)}. \tag{3.3}$$

By condition (iv), for $\eta = w_3^{-1}[\frac{w_2(\varepsilon)}{2}]$, there exists $\widetilde{T} > 0$ such that

$$\int_t^{t+\widetilde{T}} \psi(t,\eta)dt > \Omega + w_1(\rho_0)(1+M), \quad \forall t \geq t_0. \tag{3.4}$$

Let N be the first positive integer such that

$$w_3(\beta) \leq w_2(\varepsilon) + Na \tag{3.5}$$

We shall show that, for any $i = 0, 1, \cdots, N$

$$H(t, x(t)) \leq w_2(\varepsilon) + (N-i)a, \quad t \geq t_0 + t_{K^*} + i(\widetilde{T} + r), \tag{3.6}$$

It is clear that (3.6) holds for $i = 0$ since from (3.1) and (3.5)

$$H(t, x(t)) \leq w_3(\beta) \leq w_2(\varepsilon) + Na, \quad \forall t \geq t_0. \tag{3.7}$$

Suppose (3.6) holds for $i = k$, i.e.

$$H(t, x(t)) \leq w_2(\varepsilon) + (N-k)a, \quad t \geq \tau_k, \ k = 0, 1, \cdots, N-1, \tag{3.8}$$

where $\tau_k = t_0 + t_{K^*} + k(\widetilde{T} + r), \ k = 0, 1, \cdots, N-1.$
We shall show (3.6) holds for $i = k+1$, i.e.

$$H(t, x(t)) \leq w_2(\varepsilon) + (N-k-1)a, \quad t \geq \tau_{k+1}, \ k = 0, 1, \cdots, N-1. \tag{3.9}$$

Let $I_k = [\tau_k + r, \tau_{k+1}]$, we claim that there exists some $t^* \in I_k$, such that

$$H(t^*, x(t^*)) < w_2(\varepsilon) + (N-k-2)a. \tag{3.10}$$

Otherwise, for all $t \in I_k$, we have

$$H(t, x(t)) \geq w_2(\varepsilon) + (N-k-2)a. \tag{3.11}$$

From (3.2) we have $a < \frac{w_2(\varepsilon)}{2}$, noticing $k \leq N-1$, (3.1) and (3.11), we obtain

$$\frac{w_2(\varepsilon)}{2} \leq H(t, x(t)) \leq w_3(\beta), \quad \forall t \in I_k. \tag{3.12}$$

Then by (3.2), (3.8) and (3.12), we have, for any $t \in I_k$,

$$\begin{aligned} P(H(t, x(t))) &> H(t, x(t)) + 2a \\ &\geq w_2(\varepsilon) + (N-k-2)a + 2a \\ &\geq w_2(\varepsilon) + (N-k)a \\ &\geq V(t+s, x(t+s)), \qquad \forall s \in [-r, 0]. \end{aligned} \tag{3.13}$$

From condition (iii), we have, for any $t \in I_k$

$$V'(t, x(t)) \leq -F(t, h(t, x(t))) + g(t). \tag{3.14}$$

On the other hand, condition (i) and (3.12) imply, for any $t \in I_k$

$$w_3(h(t, x(t))) \geq H(t, x(t)) \geq \frac{w_2(\varepsilon)}{2},$$

i.e.

$$h(t,x(t)) \geq w_3^{-1}(\frac{w_2(\varepsilon)}{2}) = \eta > 0. \tag{3.15}$$

From (3.15) and the assumption on F, we have

$$F(t,h(t,x(t))) \geq \psi(t,\eta) \geq 0,$$

together with (3.14), we obtain

$$V'(t,x(t)) \leq -\psi(t,\eta) + g(t), \quad \forall t \in I_k. \tag{3.16}$$

Integrating (3.16) from $\tau_k + r$ to τ_{k+1}, and noticing $\tau_{k+1} = \tau_k + r + \widetilde{T}$, from (3.1), (3.4) and condition (ii) and (iv), we have

$$\begin{aligned} V(\tau_{k+1},x(\tau_{k+1})) &\leq V(\tau_k + r, x(\tau_k + r)) - \textstyle\int_{\tau_k+r}^{\tau_{k+1}} \psi(s,\eta)ds + \int_{\tau_k+r}^{\tau_{k+1}} g(s)ds \\ &\quad + \textstyle\sum_{\tau_k+r<t_k\leq\tau_{k+1}} (V(t_k,x(t_k)) - V(t_k^-,x(t_k^-))) \\ &\leq w_1(\rho_0) - \textstyle\int_{\tau_k+r}^{\tau_k+r+\widetilde{T}} \psi(s,\eta)ds + \int_0^\infty g(s)ds \\ &\quad + \textstyle\sum_{\tau_k+r<t_k\leq\tau_{k+1}} V(t_k^-,x(t_k^-))[\frac{\psi_k(V(t_k^-,x(t_k^-)))}{V(t_k^-,x(t_k^-))} - 1] \\ &\leq w_1(\rho_0) - \textstyle\int_{\tau_k+r}^{\tau_k+r+\widetilde{T}} \psi(s,\eta)ds + \Omega \\ &\quad + w_1(\rho_0)\textstyle\sum_{k=1}^{\infty}[\frac{\psi_k(w_1(\rho_0))}{w_1(\rho_0)} - 1] \\ &\leq w_1(\rho_0)(1+M) - \textstyle\int_{\tau_k+r}^{\tau_k+r+\widetilde{T}} \psi(s,\eta)ds + \Omega \\ &< 0. \end{aligned} \tag{3.17}$$

This contradicts $V(t,x(t)) \geq 0$, so (3.10) holds.

Now we prove, for all $t \geq t^*$

$$H(t,x(t)) \leq w_2(\varepsilon) + (N-k-1)a. \tag{3.18}$$

We consider two cases: $t^* \neq t_k$ for any $k \in N$ and $t^* = t_k$ for some $k \in N$.

Case 1. $t^* \neq t_k$ for any $k \in N$.

We claim (3.18) holds for all $t \geq t^*$. Otherwise, there exists some $\widehat{t} = \inf_{t\geq t^*}\{H(t,x(t)) \geq w_2(\varepsilon) + (N-k-1)a\}$, and then we have

$$H(\widehat{t},x(\widehat{t})) \geq w_2(\varepsilon) + (N-k-1)a. \tag{3.19}$$

Since $H(t^*,x(t^*)) < w_2(\varepsilon) + (N-k-2)a$ and $t^* \neq t_k$ for any $k \in N$, so $\widehat{t} > t^*$; and we have $H(t^*,x(t^*)) \leq H(t,x(t)) \leq H(\widehat{t},x(\widehat{t}))$ for all $t \in [t^*,\widehat{t}]$.

Then for all $t \in [t^*, \widehat{t}]$, we have

$$
(3.20)\qquad \begin{aligned} P(H(t,x(t))) &> H(t,x(t)) + 2a \geq H(t^*, x(t^*)) + a \\ &\geq w_2(\varepsilon) + (N-k-2)a + 2a \\ &\geq w_2(\varepsilon) + (N-k)a \\ &\geq H(t+s, x(t+s)), \qquad \forall s \in [-r, 0]. \end{aligned}
$$

By condition (iii), we have

$$
(3.21)\qquad H'(t,x(t)) \leq -F_1(t, h(t,x(t))) \leq 0.
$$

By integrating both sides of (3.21) and using (3.3) and $t^* \geq \tau_k + r > t_{K^*}$, we obtain

$$
(3.22)\qquad \begin{aligned} H(\widehat{t}, x(\widehat{t})) &\leq H(t^*, x(t^*)) + \textstyle\sum_{t^* \leq t_k} (H(t_k, x(t_k)) - H(t_k^-, x(t_k^-))) \\ &< w_2(\varepsilon) + (N-k-2)a + w_3(\beta) \textstyle\sum_{k=K^*}^{\infty} [\frac{\psi_k(w_3(\beta))}{w_3(\beta)} - 1] \\ &< w_2(\varepsilon) + (N-k-2)a + w_3(\beta) \cdot \frac{a}{w_3(\beta)} \\ &\leq w_2(\varepsilon) + (N-k-1)a, \end{aligned}
$$

which contradicts (3.19) and shows that (3.18) holds.

Case 2. $t^* = t_k$ for some $k \in N$.

In this case, we first prove that (3.18) holds for any $t \in [t^*, t_{k+1})$. Suppose not, then there exists some $t^{**} \in (t^*, t_{k+1})$ such that

$$
(3.23)\qquad H(t^{**}, x(t^{**})) = w_2(\varepsilon) + (N-k-1)a,
$$

and $H(t^*, x(t^*)) \leq H(t, x(t)) \leq H(t^{**}, x(t^{**}))$ for all $t \in [t^*, t^{**}]$. Then we have (3.20) and (3.21) hold for all $t \in [t^*, t^{**}]$. Integrating both sides of (3.21), we obtain

$$
(3.24)\qquad \begin{aligned} H(t^{**}, x(t^{**})) &\leq H(t^*, x(t^*)) \\ &\leq w_2(\varepsilon) + (N-k-2)a, \end{aligned}
$$

which contradicts (3.23) and shows (3.18) holds for any $t \in [t^*, t_{k+1})$.

Next, we shall prove that (3.18) holds for any $t \geq t_{k+1}$. Suppose not, then there exists some $t^{***} = \inf_{t \geq t_{k+1}} \{H(t, x(t)) \geq w_2(\varepsilon) + (N-k-1)a\}$, and then we have

$$
(3.25)\qquad H(t^{***}, x(t^{***})) \geq w_2(\varepsilon) + (N-k-1)a.
$$

Since $H(t^*, x(t^*)) < w_2(\varepsilon) + (N-k-2)a$ and (3.18) holds for any $t \in [t^*, t_{k+1})$, so $\widehat{t} > t^*$; and we have $H(t^*, x(t^*)) \leq H(t, x(t)) \leq H(\widehat{t}, x(\widehat{t}))$ for all $t \in [t^*, t^{***}]$. Then using similar way as that in case 1, we can prove (3.20) and (3.21) hold for all $t \in [t^*, t^{***}]$. And then by integrating (3.21) from t^* to t^{***}, we can obtain $H(t^{***}, x(t^{***})) < w_2(\varepsilon) + (N-k-1)a$ similarly as that in (3.22). This contradicts (3.25) and shows that (3.18) holds.

Thus we know (3.18) holds in both cases, and hence (3.9) is true since $t^* \leq \tau_{k+1}$.

So by induction, (3.6) holds for $i = 0, 1, \cdots, N$. Let $i = N$ in (3.6), we have

$$
w_2(h(t, x(t))) \leq H(t, x(t)) \leq w_2(\varepsilon), \quad \forall t \geq \tau_N = t_0 + T^*,
$$

i.e.

$$h(t, x(t)) \leq \varepsilon, \quad \forall t \geq \tau_N = t_0 + T^*,$$

where $T^* = t_{K^*} + N(\widetilde{T} + r)$ is independent of t_0. The proof is complete.

Corollary 3.1. *If condition (iii) and (iv) of Theorem 3.1 are replaced by (iii') and (iv') respectively,*

(iii') *for any $\lambda_i \geq 0 (i = 1, 2)$, there exist constants $r,\ T^* > 0$ such that for any solution $x(t)$ of (2.1) and $t \geq T^*$*

$$\begin{aligned} V'(t, x(t)) &\leq -b(t)\ [\ w_4(h(t, x(t))) - \lambda_1\ ] + g(t), \\ H'(t, x(t)) &\leq -c(t)\ [\ w_5(h(t, x(t))) - \lambda_2\ ], \end{aligned}$$

whenever $h(t, x(t)) < \rho$, and $P(V(t, x(t))) > V(t + s, x(t + s)),\ -r \leq s \leq 0$, where $P \in C(R_+, R_+),\ P(s) > s$ for $s > 0$, $b(t),\ c(t) \geq 0$;

(iv') $\lim_{T\to\infty} \inf_{t\geq 0} \int_t^{t+T} b(s)ds = \infty$.

Then the result is still true.

Proof. Let $\lambda_1 = \frac{w_4(\sigma)}{2}$ for any $\sigma > 0$, we have

$$b(t)\ [\ w_4(\sigma) - \lambda_1\ ]\ = b(t)\frac{w_4(\sigma)}{2} \triangleq \widetilde{\psi}(t, \sigma), \ \ \forall t \geq T^*, \tag{3.26}$$

so condition (iii) of Theorem 3.1 can be rewritten so that there exist constants $r,\ T^* > 0$ such that for any solution $x(t)$ of (2.1)

$$\begin{aligned} V'(t, x(t)) &\leq -b(t)\ [\ w_4(h(t, x(t))) - \lambda_1\ ]\ + g(t) \\ &\triangleq\ -F_1(t, h(t, x(t)))\ + g(t), \quad t \geq T^*, \end{aligned} \tag{3.27}$$

whenever $h(t, x(t)) < \rho$ and $P(V(t, x(t))) > V(t+s, x(t+s)),\ -r \leq s \leq 0$, where $P \in C(R_+, R_+),\ P(s) > s$ for $s > 0$ and $F(t, h(t, x(t))) \geq \widetilde{\psi}(t, \sigma) \geq 0$ for $h(t, x(t)) \geq \sigma > 0$. Moreover, let $F_1(t, h(t, x(t))) = c(t)\ [\ w_5(h(t, x(t))) - \lambda_2\ ]$, then $F_1(t, h(t, x(t))) \geq 0$. Together with condition (iv'), we know condition (iv) of Theorem 3.1 holds. This completes the proof.

Remark 3.1. *In Corollary 3.1, if we let $h_0(t, x(t)) = \|x(t)\|_r,\ h(t, x(t)) = \|x(t)\|$, where $\|\cdot\|$ is any norm in R^n, and let $J_k(x) \equiv x$, then we can get the same UAS result in Theorem 2.1 in reference* [12].

Corollary 3.2. *If condition (ii) of Theorem 3.1 is replaced by (ii'),*

(ii') *$V(t_k, J_k(x)) \leq (1+b_k)V(t_k^-, x)$ and $H(t_k, J_k(x)) \leq (1+b_k)H(t_k^-, x)$, if $h(t, x) < \rho$, where $b_k > 0$ and $\sum_{k=1}^{\infty} b_k < \infty$.*

Then the result is still true.

Proof. Let $\psi_k(s) = (1 + b_k)s$ in condition (ii) of Theorem 3.1, together with $b_k > 0$ and $\sum_{k=1}^{\infty} b_k < \infty$, then condition (ii) of Theorem 3.1 holds.

4. APPLICATION TO LOTKA-VOLTERRA SYSTEMS WITH TIME DELAY AND IMPULSIVE EFFECTS

Functional differential equations are frequently used to model population dynamics. The Lotka-Volterra equation for predator-pray problems or that of competing species is often considered([2, 4, 12]). When population levels repeatedly undergo changes of relatively short duration(due, for instance, to stocking or harvesting of species), the these events may be more suitably modelled by an impulsive functional differential equation (see [8] and references therein).

Consider the following Lotka-Volterra system subject to impulsive effects

$$x_i'(t) = b_i(x_i(t))\big\{r_i(t) - a_i(t)x_i(t) + \sum_{j=1}^{n}\int_{-\infty}^{t} x_j(s)d\mu_{ij}(t,s)\big\},\ t \neq t_k, \tag{4.1a}$$

$$x_i(t_k) = c_{ik}x_i(t_k^-) + (1-c_{ik})x_i^*,\ i = 1,\cdots,n. \tag{4.1b}$$

$$x(\theta) = \phi(\theta), \quad \theta \in (-\infty, 0]., \tag{4.1c}$$

where $x^* = (x_1^*, x_2^*, \cdots, x_n^*)$ is assumed to be a positive equilibrium of system (4.1a) and (4.1c), and initial functions satisfy

$$\phi_i(\theta) \geq 0, \quad \phi_i(0) > 0, \quad \text{for } i = 1,2,\cdots,n. \tag{4.2}$$

Suppose for $i,\ j = 1,2,\cdots,n,\ k \in N$, the following conditions hold

(A1). Constants $c_{ik} \in [0,1]$ and functions $b_i \in K_0$ and for any $0 < \beta \ll 1$, $\int_0^\beta \frac{ds}{b_i(s)} = +\infty$;

(A2). $r_i(t) \geq 0$ and $a_i(t) \geq 0$ are continuous functions;

(A3). $\mu_{ij}(t,s)$ have bounded variation for any $t \in R$ and $s \leq t$, and satisfy

$$\int_{-\infty}^{u} |d\mu_{ij}(t,s)| \leq a_i(t)\hat{\mu}_{ij}(t,u),$$

where $\hat{\mu}_{ij}(t,u)(u \leq t)$ are nondecreasing with respect to u, and there exist constants $\gamma_{ij} \geq 0$ with $\gamma_{ii} < 1$ such that $\hat{\mu}_{ij}(t,t) \leq \gamma_{ij}$, and for any $\varepsilon > 0$, there exists constant $h > 0$ such that $\hat{\mu}_{ij}(t, t-h) \leq \varepsilon$, $\forall t \geq 0$.

Remark 4.1. *From Lemma 3.1([12]), we know the solutions of (4.1a) and (4.1c) are positive in their maximal existence intervals. So the solutions of (4.1) are positive in their maximal existence intervals, since* $c_{ik} \in [0,1]$ *for any* $i = 1,\cdots,n,\ k \in N$.

Theorem 4.1. *Assume conditions* (A1)-(A3) *hold, and*

(i). $b_i(s)$ *are nondecreasing and for any* $i = 1,\cdots,n,\ t \in R$

$$\lim_{p\to+\infty}\int_t^{t+p} a_i(s)ds = +\infty \tag{4.3}$$

(ii). Γ_1 *is an M-matrix, where*

$$\Gamma_1 = \begin{bmatrix} 1-\gamma_{11} & -\gamma_{12} & \cdots & -\gamma_{1n} \\ -\gamma_{21} & 1-\gamma_{22} & \cdots & -\gamma_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ -\gamma_{n1} & -\gamma_{n2} & \cdots & 1-\gamma_{nn} \end{bmatrix}$$

then x^* *is uniformly asymptotically stable.*

Proof. Rewrite system (4.1a) and (4.1b), we obtain

$$y_i'(t) = b_i(y_i(t)+x_i^*)\Big\{-a_i(t)y_i(t) + \sum_{j=1}^{n}\int_{-\infty}^{t} y_j(s)d\mu_{ij}(t,s)\Big\},\ t \neq t_k, \tag{4.4a}$$

$$y_i(t_k) = c_{ik}y_i(t_k^-),\ i = 1,\cdots,n. \tag{4.4b}$$

where $y_i(t) = x_i(t) - x_i^*, i = 1,\cdots,n$. Since Γ_1 is an M-matrix, there exist positive constants $d_i,\ i = 1,\cdots,n$, such that

$$d_i(1-\gamma_{ii}) > \sum_{i\neq j}^{n} d_j\gamma_{ij},\quad i = 1,\cdots,n. \tag{4.5}$$

Choose

$$\begin{aligned} H(y(t)) &= \max\{d_i^{-1}|y_i(t)|:\ 1\le i\le n\}, \\ N_H &= \{i\in\{1,2,\cdots,n\}:\ H(y(t)) = d_i^{-1}|y_i(t)|,\ t\ge 0\}. \end{aligned}$$

For any $i \in N_H$, using (4.5), we calculate $H'(y(t))$

$$\begin{aligned} H'(y(t)) &\le b_i(x_i(t))\big\{-a_i(t)|y_i(t)| + \textstyle\sum_{j=1}^{n}\int_{-\infty}^{t}|y_j(s)||d\mu_{ij}(t,s)|\big\} \\ &\le -b_i(x_i(t))d_i^{-1}\big\{d_i(1-\gamma_{ii}) - \textstyle\sum_{i\neq j}^{n} d_j\gamma_{ij}\big\}a_i(t)H(y(t)) \\ &\le 0 \end{aligned} \tag{4.6}$$

whenever $H(s, y(s)) \le H(t, y(t))$ for $s \le t$.

And

$$\begin{aligned} H(y(t_k)) &= \max\{d_i^{-1}|y_i(t_k)|:\ 1\le i\le n\} \\ &= \max\{d_i^{-1}c_{ik}|y_i(t_k^-)|:\ 1\le i\le n\} \\ &\le H(y(t_k^-)). \end{aligned}$$

Thus we know from Theorem 2.1([11]) that the trivial solution of (4.4) is uniformly stable.

Now choose $h_0(t, y(t)) = \|y(t)\|_\infty = \sup_{-\infty<s\le 0}\Big\{\max_{1\le i\le n}\{d_i^{-1}|y_i(t+s)|\}\Big\}$, $h(t, y(t)) = \|y(t)\|_n = \max_{1\le i\le n}\{d_i^{-1}|y_i(t)|\}$, then for any given $\varepsilon > 0$, there exists $\delta = \delta(\varepsilon) > 0$ such that $\|\phi\|_\infty \le \delta$ implies $|y_i(t)| \le \varepsilon$ for $t \ge t_0,\ i = 1, 2,\cdots,n$.

Then for $t \ge t_0$, let

$$V(y(t)) = \max_{1\le i\le n}\Big\{d_i^{-1}V_i(t),\ \text{where } V_i(t) = \int_0^{|y_i|}\frac{du}{b_i(x_i^* + sgn(y_i)u)}\Big\}, \tag{4.7}$$

and

$$N_V = \{i \in \{1, 2, \cdots, n\} : V(y(t)) = d_i^{-1} V_i(t), \ t \geq 0\}.$$

We have $\frac{1}{b_i(x_i^* + sgn(y_i)u)} > 0$ since

$$x_i^* + sgn(y_i)u = \begin{cases} x_i^* + u \geq x_i^* > 0, & \text{if } y_i \geq 0, \\ x_i^* - u \geq x_i^* + y_i = x_i > 0, & \text{if } y_i < 0, \end{cases}$$

so $V(y(t)) \geq 0$.

And we have $V(y(t_k)) \leq V(y(t_k^-))$ since

$$V_i(y(t_k)) \ = \ \int_0^{c_{ik}|y_i(t_k^-)|} \frac{du}{b_i(x_i^* + sgn(y_i)u)} \ \leq \ V_i(y(t_k^-))$$

in view of $c_{ik} \in [0, 1]$.

Moreover, choose $\rho > 0$ such that $\rho < \min_{1 \leq i \leq n}\{d_i^{-1} x_i^*\}$, then there exists $\eta > 0$ such that

$$\rho \leq \min_{1 \leq i \leq n} \{d_i^{-1}(x_i^* - \eta)\}.$$

Thus, together with (4.7), we have, for $h(t, y(t)) = \|y(t)\|_n < \rho$

$$d_i^{-1}(x_i^* - \eta) \geq \rho > \|y(t)\|_n \geq d_i^{-1}|y_i(t)|,$$

i.e. $x_i^* - |y_i(t)| \geq \eta$. Since $b_i(s)$ are nondecreasing, we have

$$\frac{1}{b_i(x_i^* + sgn(y_i)u)} \leq \frac{1}{b_i(x_i^* - |y_i(t)|)} \leq \frac{1}{b_i(\eta)}, \ \forall u \in (0, |y_i(t)|).$$

which implies $V(y(t)) \leq \max_{1 \leq i \leq n} \left\{\frac{d_i^{-1}}{b_i(\eta)}|y_i(t)|\right\}$ when $h(t, y(t)) < \rho$, thus condition (i) of Corollary 3.1 holds.

By assumption $(A3)$, for any given $\sigma_1 > 0$, there exists $h > 0$ such that

$$\varepsilon \sum_{j=1}^{n} \hat{\mu}_{ij}(t, t - h) \leq \sigma_1, \ \text{ for } i = 1, 2, \cdots, n. \tag{4.8}$$

By (4.5), there exists $\rho_1 > 1$ such that

$$d_i > \rho_1 \sum_{j=1}^{n} d_j \gamma_{ij}, \quad i = 1, 2, \cdots, n. \tag{4.9}$$

By assumption $(A2)$ and inequalities (4.8) and (4.9), for any $i \in N_V$, we have

$$\begin{aligned} V'(y(t)) &= -d_i^{-1} a_i(t)|y_i(t)| + \textstyle\sum_{j=1}^{n} d_i^{-1} \left\{ \int_{t-h}^{t} + \int_{-\infty}^{t-h} \right\} |y_j(s) d\mu_{ij}(t, s)| \\ &\leq -d_i^{-1} a_i(t) \left\{ H(y(t))[d_i - \rho_1 \textstyle\sum_{j=1}^{n} d_j \gamma_{ij}] - \sigma_1 \right\}, \end{aligned}$$

whenever $H(y(s)) \leq \rho_1 H(y(t))$ for $s \in [t - h, t]$. Similarly, we obtain

$$H'(y(t)) \leq -b_i(x_i(t)) a_i(t) d_i^{-1} \left\{ H(y(t))[d_i - \rho_1 \sum_{j=1}^{n} d_j \gamma_{ij}] - \sigma_1 \right\},$$

whenever $H(y(s)) \leq \rho_1 H(y(t))$ for $s \in [t-h, t]$. Choose $P(s) = \rho_1 s$, then all conditions of Corollary 3.1 are satisfied, and hence the equilibrium x^* of system (4.1) is uniformly asymptotically stable.

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