QUENCHING FOR DEGENERATE PARABOLIC PROBLEMS WITH NONLOCAL BOUNDARY CONDITIONS

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ABSTRACT. Let q be a nonnegative real number, and a and T be positive constants. This article studies the following degenerate parabolic problem:

$$x^q u_t - u_{xx} = G(u) \text{ in } (0,a) \times (0,T],$$

where G is a nonnegative function in the form of either $f(u(x,t))$, or $\int_0^a h(x,t)\, f(u(x,t))dx$ for some positive, bounded and continuous function h with $f > 0$, $f' > 0$, $f'' \geq 0$, and $\lim_{u \to 1^-} f(u) = \infty$. It is subject to the initial condition,

$$u(x,0) = 0 \text{ on } [0,a],$$

and the boundary conditions,

$$u(0,t) = \int_0^a M(x)\,|u\,(x,t)|^p\,dx,\; u\,(a,t) = \int_0^a N\,(x)\,|u\,(x,t)|^r\,dx, t > 0,$$

where p and r are constants greater than or equal to 1, and M and N are given nonnegative functions. Existence, uniqueness and criteria for quenching and non-quenching are studied.

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1. INTRODUCTION

Let a, p, r and T be positive constants with $p \geq 1$ and $r \geq 1$, $D = (0,a)$, $\bar{D} = [0,a]$, $\Omega = D \times (0,T]$, $\bar{\Omega} = \bar{D} \times [0,T]$, and $Lu = x^q u_t - u_{xx}$, where q is a nonnegative real number. Let us consider the following initial nonlocal boundary-value problem:

$$Lu = G(u) \text{ in } \Omega, \tag{1.1}$$

$$u(x,0) = 0 \text{ on } \bar{D}, \tag{1.2}$$

$$\begin{cases} u(0,t) = \int_0^a M(x)|u(x,t)|^p dx, \\ u(a,t) = \int_0^a N(x)|u(x,t)|^r dx, \quad 0 < t \leq T, \end{cases} \tag{1.3}$$

where $M(x) \geq 0$, $\int_0^a M(x)dx \leq 1$, $N(x) \geq 0$, and $\int_0^a N(x)dx \leq 1$. Here, $G\,(u)$ is in the form of either $f(u(x,t))$, or $\int_0^a h(x,t)f(u(x,t))dx$, where $f > 0$, $f' > 0$, $f'' \geq 0$,

$\lim_{u\to 1^-} f(u) = \infty$, and h is positive, bounded and continuous. The solution u is said to quench if $\lim_{t\to T^-} \max_{\bar{D}} u(x,t) = 1$. If $\int_0^a M(x)dx = 0$ and $\int_0^a N(x)dx = 0$, then $M(x) = 0 = N(x)$ a.e. on $\bar{D}$, and we have the first boundary conditions $u(0,t) = 0 = u(a,t)$. These boundary conditions with $G(u) = f(u)$ was studied by Chan and Kong in [1] for the case $\int_0^1 f(s)\,ds < \infty$, and in [2] for the case $\int_0^1 f(s)\,ds = \infty$. In the sequel, we assume that $\int_0^a M(x)dx$ and $\int_0^a N(x)dx$ are positive. We note that a quenching problem involving a homogeneous heat equation subject to a nonlocal Neumann boundary condition was studied by Roberts and Olmstead [8].

In section 2, we show that the problem (1.1)–(1.3) has a unique classical solution. In section 3, we give a criterion for quenching to occur, and conditions for global existence.

2. UNIQUENESS AND EXISTENCE

Since $M(x)$ and $N(x)$ are nonnegative, if u is a solution of the problem (1.1)-(1.3), then $u(0,t)$ and $u(a,t)$ are nonnegative. Because $Lu > 0$ in Ω, it follows from the strong maximum principle (cf. Friedman [4, p. 39]) that $u > 0$ in Ω.

We now prove a comparison result. Let $B(v(x,t))$ denote $K(x,t)v(x,t)$ or $\int_0^a K(x,t)v(x,t)dx$ for some bounded nonnegative function $K(x,t)$. Also, let $K_1(x,t)$ and $K_2(x,t)$ be some nontrivial, nonnegative, bounded and continuous functions.

Lemma 2.1. *If $Lv(x,t) > B(v(x,t))$ in Ω, $v(x,0) > 0$ on $\bar{D}$,*

$$v(0,t) > \int_0^a K_1(x,t)v(x,t)dx,\ v(a,t) > \int_0^a K_2(x,t)v(x,t)dx,\ 0 < t \le T,$$

then $v(x,t) > 0$ on $\bar{\Omega}$.

Proof. Suppose that $v(x,t) \le 0$ somewhere on $\bar{\Omega}$. Since $v(x,0) > 0$, there are $t_1 > 0$ and $x_1 \in \bar{D}$ such that $v(x_1,t_1) = 0$ and $v(x,t) > 0$ for $(x,t) \in \bar{D} \times [0,t_1)$. If $x_1 \in D$, then $v_t(x_1,t_1) \le 0$ and $v_{xx}(x_1,t_1) \ge 0$. This implies $Lv(x_1,t_1) \le 0$. Since it is given that $Lv(x_1,t_1) - Bv(x_1,t_1) > 0$, we have a contradiction. Therefore either $x_1 = 0$ or $x_1 = a$. But in either case, we have $0 > \int_0^a K_1(x,t_1)v(x,t_1)dx \ge 0$, or $0 > \int_0^a K_2(x,t_1)v(x,t_1)dx \ge 0$. Thus, $v > 0$ on $\bar{\Omega}$. □

Theorem 2.2. *If*

$$Lv \ge B(v) \ \ in\ \Omega,$$

$$v(x,0) \ge 0 \ \ on\ \bar{D},$$

$$v(0,t) \ge \int_0^a K_1(x,t)v(x,t)dx,\ v(a,t) \ge \int_0^a K_2(x,t)v(x,t)dx,\ 0 < t \le T,$$

then $v \ge 0$ on $\bar{\Omega}$.

Proof. Let $\bar{M} = \max_{\bar{D}}\{K_1(x,t), K_2(x,t)\}$. Let us choose a natural number $\bar{k}$ such that

$$1 - \left(\frac{2\bar{M}}{2\bar{k}+1}\right)\left(\frac{a}{2}\right) > 0,$$

and a positive real number A such that

$$A\left(\frac{a}{2}\right)^{2\bar{k}}\left[1 - \frac{2\bar{M}}{2\bar{k}+1}\left(\frac{a}{2}\right)\right] > \frac{3}{5}\bar{M}a^{5/2} + \gamma(\bar{M}a - 1), \tag{2.1}$$

where γ is an arbitrarily fixed positive constant.

For a fixed positive real number η, let

$$w(x,t) = v(x,t) + \eta g(x)e^{\kappa t},$$

where

$$g(x) = A\left(x - \frac{a}{2}\right)^{2\bar{k}} + a^{3/2} - x^{3/2} + \gamma,$$

and κ is some positive constant to be determined. We have

$$g''(x) = 2\bar{k}(2\bar{k}-1)A\left(x - \frac{a}{2}\right)^{2\bar{k}-2} - \frac{3}{4}x^{-1/2},$$

$$(L-B)w = (L-B)v + x^q\kappa\eta g(x)e^{\kappa t} - \eta g''(x)e^{\kappa t} - B(\eta g(x)e^{\kappa t}).$$

Since in $g''(x)$, $x^{-1/2}$ is unbounded at $x = 0$, there exists some real number $\delta \in D$ such that $-\eta g''(x)e^{\kappa t} - B(\eta g(x)e^{\kappa t}) > 0$ for $0 < x \le \delta$. For $\delta < x < a$, let us choose κ such that

$$\delta^q\kappa\eta g(x)e^{\kappa t} - \eta g''(x)e^{\kappa t} - B(\eta g(x)e^{\kappa t}) > 0.$$

Then,

$$Lw > B(w) \text{ in } \Omega.$$

Also, $w(x,0) = v(x,0) + \eta g(x) > 0$ on $\bar{D}$. At $x = 0$, we have

$$g(0) = A\left(\frac{a}{2}\right)^{2\bar{k}} + a^{3/2} + \gamma,$$

$$\int_0^a K_1(x,t)\eta g(x)e^{\kappa t}dx \le \eta\bar{M}e^{\kappa t}\left[\frac{2A}{2\bar{k}+1}\left(\frac{a}{2}\right)^{2\bar{k}+1} + \frac{3}{5}a^{5/2} + \gamma a\right].$$

These give

$$w(0,t) \ge \int_0^a K_1(x,t)v(x,t)dx + \eta\left[A\left(\frac{a}{2}\right)^{2\bar{k}} + a^{3/2} + \gamma\right]e^{\kappa t}.$$

From (2.1),

$$A\left(\frac{a}{2}\right)^{2\bar{k}} + \gamma > \bar{M}\left[\frac{2A}{2\bar{k}+1}\left(\frac{a}{2}\right)^{2\bar{k}+1} + \frac{3}{5}a^{5/2} + \gamma a\right].$$

Therefore,

$$w(0,t) > \int_0^a K_1(x,t)w(x,t)dx.$$

Similarly,

$$w(a,t) > \int_0^a K_2(x,t)w(x,t)dx.$$

By Lemma 2.1, $w(x,t) > 0$ on $\bar{\Omega}$. As $\eta \to 0$, we obtain $v(x,t) \ge 0$. □

We now prove a uniqueness result.

Theorem 2.3. *The problem* (1.1)–(1.3) *has at most one solution* u.

Proof. Let u and v be two solutions of the problem (1.1)-(1.3), and $w = u - v$. By the mean value theorem,

$$Lw = G'(\xi)(u - v),$$

where ξ is a function between u and v. We have $w(x, 0) = 0$. Using the mean value theorem, we have for some functions ζ_1 and ζ_2,

$$w(0, t) = \int_0^a M(x)p\zeta_1^{p-1}(x, t)w(x, t)dx,$$

$$w(a, t) = \int_0^a N(x)r\zeta_2^{r-1}(x, t)w(x, t)dx.$$

By Theorem 2.2, $w(x, t) = 0$. This contradiction proves the theorem. □

Theorem 2.4. *The solution* u *is nondecreasing with respect to* t.

Proof. Let $0 < h < T$, and $w(x, t) = u(x, t + h) - u(x, t)$. Then,

$$Lw(x, t) = G(u(x, t + h)) - G(u(x, t)) = G'(\xi)w(x, t),$$

where ξ lies between $u(x, t + h)$ and $u(x, t)$. Since $u(x, 0) = 0$ and $u(x, t) > 0$ in Ω, we have $w(x, 0) > 0$. Using the mean value theorem, we have for some functions ξ_1 and ξ_2, $w(0, t) = \int_0^a M(x)p\xi_1^{p-1}w(x, t)dx$ and $w(a, t) = \int_0^a N(x)r\xi_2^{r-1}w(x, t)dx$. By Theorem 2.2, $w \geq 0$ on $\bar{\Omega}$. Hence $u(x, t)$ is nondecreasing with respect to t. □

Let k be a positive integer such that

$$\left(\frac{a}{2}\right)\left(\frac{2\max M(x)}{2k + 1}\right) < \frac{1}{8}.$$

Let c_1 and c_2 be positive real numbers such that

$$\max M(x)\left(\frac{2}{3}a^{\frac{3}{2}}\right)c_1 < \frac{1}{16},\ c_1a^{\frac{1}{2}} < \frac{1}{2},\ \frac{1}{4} < c_2\left(\frac{a}{2}\right)^{2k} < \frac{1}{2}.$$

Then, $c_1a^{1/2} + c_2(a/2)^{2k} < 1$. We consider the function,

$$\tilde{v}(x, t) = \left[c_1x^{\frac{1}{2}} + c_2\left(x - \frac{a}{2}\right)^{2k}\right]e^{\tilde{K}t-1},$$

where $\tilde{K}$ is a positive constant to be determined. Since

$$\tilde{v}_{xx} = \left[-\frac{c_1}{4}x^{-\frac{3}{2}} + (2k)(2k - 1)c_2\left(x - \frac{a}{2}\right)^{2k-2}\right]e^{\tilde{K}t-1}$$

is unbounded at $x = 0$, there exists some real number $\delta \in D$ such that $\tilde{v}_{xx} + G(\tilde{v}) \leq 0$ for $0 < x \leq \delta$. This can be achieved by choosing δ satisfying

$$\left[-\frac{c_1}{4}x^{-\frac{3}{2}} + (2k)(2k-1)c_2\left(x - \frac{a}{2}\right)^{2k-2}\right]e^{\tilde{K}t-1}$$
$$+G\left(\left[c_1\delta^{\frac{1}{2}} + c_2\left(\frac{a}{2}\right)^{2k}\right]e^{\tilde{K}t-1}\right) \leq 0$$

for $0 < x \leq \delta$. For $\delta < x < a$, let us choose $\tilde{K}$ such that $x^q\tilde{v}_t(x,0) > \tilde{v}_{xx}(x,0) + G(\tilde{v}(x,0))$. This can be accomplished by choosing $\tilde{K}$ satisfying

$$\tilde{K}\delta^q\left(c_1\delta^{\frac{1}{2}}\right)e^{-1} > \left[-\frac{c_1}{4}\delta^{-\frac{3}{2}} + (2k)(2k-1)c_2\left(\frac{a}{2}\right)^{2k-2}\right]e^{-1}$$
$$+ G\left(\left[c_1a^{\frac{1}{2}} + c_2\left(\frac{a}{2}\right)^{2k}\right]e^{-1}\right).$$

There exists some $\hat{t}$ (> 0) such that $L\tilde{v}(x,t) \geq G(\tilde{v}(x,t))$ for $\delta < x < a$, $0 < t < \hat{t}$, and $\tilde{v}(x,t) < 1$. We now have

$$L\tilde{v} \geq G(\tilde{v}) \text{ and } \tilde{v} < 1 \text{ in } D \times (0, \hat{t}),$$

$$\tilde{v}(x, 0) > 0 \text{ on } \bar{D},$$

$$\begin{aligned}
\tilde{v}(0,t) &= c_2\left(\frac{a}{2}\right)^{2k}e^{\tilde{K}t-1} > \frac{1}{4}e^{\tilde{K}t-1} > \left(\frac{1}{16} + \frac{1}{2}\cdot\frac{1}{8}\right)e^{\tilde{K}t-1} \\
&> \max M(x)\left(\frac{2}{3}a^{\frac{3}{2}}\right)c_1e^{\tilde{K}t-1} + c_2\left(\frac{a}{2}\right)^{2k+1}\left(\frac{2\max M(x)}{2k+1}\right)e^{\tilde{K}t-1} \\
&= \max M(x)\left[\left(\frac{2}{3}a^{\frac{3}{2}}\right)c_1 + c_2\left(\frac{a}{2}\right)^{2k+1}\left(\frac{2}{2k+1}\right)\right]e^{\tilde{K}t-1} \\
&= \max M(x)\int_0^a\left[c_1x^{\frac{1}{2}} + c_2\left(x - \frac{a}{2}\right)^{2k}\right]e^{\tilde{K}t-1}dx \\
&\geq \int_0^a M(x)\tilde{v}(x,t)dx \geq \int_0^a M(x)\tilde{v}^p(x,t)dx, \\
\tilde{v}(a,t) &= \left[c_1a^{\frac{1}{2}} + c_2\left(\frac{a}{2}\right)^{2k}\right]e^{\tilde{K}t-1} \geq \int_0^a N(x)\tilde{v}(x,t)dx \geq \int_0^a N(x)\tilde{v}^r(x,t)dx.
\end{aligned}$$

An argument similar to that in the proof of Theorem 2.4 shows that $\tilde{v} \geq u$ on $\bar{D}\times[0, \hat{t}]$.

We now show existence of the solution. Let $\Omega_{\hat{t}} = D \times (0, \hat{t}]$, and $\bar{\Omega}_{\hat{t}}$ be its closure.

Theorem 2.5. *The problem* (1.1)-(1.3) *has a unique solution* $u \in C(\bar{\Omega}_{\hat{t}}) \cap C^{2,1}(\Omega_{\hat{t}})$.

Proof. Let $u_0(x,t) \equiv 0$. For $n \geq 1$, let u_n be the solution of the problem,

$$Lu_n = G(u_{n-1}) \text{ in } \Omega_{\hat{t}},$$

$$u_n(x, 0) = 0 \text{ on } \bar{D},$$

$$u_n(0,t) = \int_0^a M(x)u_{n-1}^p(x,t)dx,\ u_n(a,t) = \int_0^a N(x)u_{n-1}^r(x,t)dx,\ 0 < t \leq \hat{t}.$$

Since $\tilde{v} > 0$, we have $\tilde{v} > u_0$ in $\Omega_{\hat{t}}$. Suppose that $\tilde{v} \geq u_n$ in $\Omega_{\hat{t}}$. Then,

$$L(\tilde{v} - u_{n+1}) \geq G(\tilde{v}) - G(u_n) \geq 0 \text{ in } \Omega_{\hat{t}},$$

$$(\tilde{v} - u_{n+1})(x, 0) > 0 \text{ on } \bar{D},$$

$$(\tilde{v} - u_{n+1})(0, t) \geq \int_0^a M(x)(\tilde{v}^p(x, t) - u_n^p(x, t))dx \geq 0,\ 0 < t \leq \hat{t},$$

$$(\tilde{v} - u_{n+1})(a, t) \geq \int_0^a N(x)(\tilde{v}^r(x, t) - u_n^r(x, t))dx \geq 0,\ 0 < t \leq \hat{t}.$$

By Theorem 2.2, $\tilde{v} - u_{n+1} \geq 0$ in $\Omega_{\hat{t}}$. It follows from the principle of mathematical induction that for any nonnegative integer n, $\tilde{v}(x, t) \geq u_n(x, t)$ for (x, t) in $\Omega_{\hat{t}}$. By using an argument similar to the proof of Theorem 2.4 and the principle of mathematical induction, we have $u_n(x, t) \geq u_{n-1}(x, t)$ in $\Omega_{\hat{t}}$, and $u_n(x, t)$ is nondecreasing with respect to t.

We now prove that $u_n(x, t)$ exists.

For $n = 1$, we consider the problem

$$\begin{cases} Lu_1 = G(0) \text{ in } \Omega_{\hat{t}}, \\ u_1(x, 0) = 0 \text{ on } \bar{D},\ u_1(0, t) = 0 = u_1(a, t) \quad \text{for } 0 < t \leq \hat{t}. \end{cases} \tag{2.2}$$

To show that the problem (2.2) has a solution, we let $\omega_\delta = (\delta, a) \times (0, \hat{t}]$, where $\delta \in (0, a)$, and $\bar{\omega}_\delta$ be its closure. We consider the problem,

$$Lu_{1\delta} = G(0) \text{ in } \omega_\delta,$$

$$u_{1\delta}(x, 0) = 0 \text{ on } \bar{D},\ u_{1\delta}(\delta, t) = 0 = u_{1\delta}(a, t) \text{ for } 0 < t \leq \hat{t}.$$

By Theorem 4.2.1 of Ladde, Lakshmikantham and Vatsala [5, pp. 139–142], the problem has a solution $u_{1\delta} \in C^{2+\alpha, 1+\alpha/2}(\bar{\omega}_\delta)$ for some $\alpha \in (0, 1)$. By Theorem 2.2, $u_{1\delta_1} < u_{1\delta_2}$ in ω_{δ_1} if $\delta_1 > \delta_2$. Since $\tilde{v}(x, t) \geq u_{1\delta}(x, t)$, it follows that $\lim_{\delta\to 0} u_{1\delta}$ exists. Let $\lim_{\delta\to 0} u_{1\delta}(x, t)$ be denoted by $u_1(x, t)$.

We are now going to show that $u_1 \in C(\bar{\Omega}_{\hat{t}}) \cap C^{2,1}(\Omega_{\hat{t}})$. For any $(\check{x}_1, \check{t}_1) \in \Omega_{\hat{t}}$, there is a set $Q_1 = [\check{b}_1, \check{b}_2] \times [\check{t}_2, \check{t}_3] \subset \bar{\Omega}_{\hat{t}}$, where $\check{b}_1$, $\check{b}_2$, $\check{t}_2$ and $\check{t}_3$ are positive numbers such that $\check{b}_1 < \check{x}_1 < \check{b}_2 < a$ and $\check{t}_2 < \check{t}_1 \leq \check{t}_3$. Since $1 > \tilde{v}(x, t) \geq u_{1\delta}(x, t)$, there is some constant $\check{p} > 1$ and some positive constants $\check{k}_1$, $\check{k}_2$ such that
(i) $||u_{1\delta}||_{L^{\check{p}}(Q_1)} \leq ||\tilde{v}||_{L^{\check{p}}(Q_1)} \leq \check{k}_1$,

(ii) $\|x^{-q}G(0)\|_{L^{\check{p}}(Q_1)} \leq \check{b}_1^{-q} \|G(\tilde{v})\|_{L^{\check{p}}(Q_1)} \leq \check{k}_2$.
By Ladyženskaja, Solonnikov and Ural′ceva [6, pp. 341–342], $u_{1\delta} \in W^{2,1}_{\check{p}}(Q_1)$. By the embedding theorems there [6, pp. 61 and 80], $W^{2,1}_{\check{p}}(Q_1) \hookrightarrow H^{\check{\alpha}, \check{\alpha}/2}(Q_1)$ by choosing

$\check{p} > 2/(1-\check{\alpha})$ with $\check{\alpha} \in (0,1)$. Then, $||u_{1\delta}||_{H^{\check{\alpha},\check{\alpha}/2}(Q_1)} \leq \check{k}_3$ for some constant $\check{k}_3$. Now,

$$\begin{aligned}\left\|x^{-q}G(0)\right\|_{H^{\check{\alpha},\check{\alpha}/2}(Q_1)} &\leq \check{b}_1^{-q}G(0) + \sup_{\substack{(x_1,t)\in Q_1\\(x_2,t)\in Q_1}} \frac{\left|x_1^{-q}G(0) - x_2^{-q}G(0)\right|}{|x_1-x_2|^{\check{\alpha}}}\\ &\leq \check{b}_1^{-q}G(0) + q\check{b}_1^{-(q+1)}G(0)\sup|x_1-x_2|^{1-\check{\alpha}}\\ &\leq \check{k}_4 \text{ for some constant } \check{k}_4.\end{aligned}$$

By Theorem 4.10.1 of Ladyženskaja, Solonnikov and Ural′ceva [6, pp. 351–352], we have

$$\|u_{1\delta}\|_{H^{2+\check{\alpha},1+\check{\alpha}/2}(Q_1)} \leq \check{K}$$

for some constant $\check{K}$ which is independent of δ. This implies that $u_{1\delta}$, $(u_{1\delta})_t$, $(u_{1\delta})_x$ and $(u_{1\delta})_{xx}$ are equicontinuous in Q_1. By the Ascoli-Arzela theorem,

$$\|u_1\|_{H^{2+\check{\alpha},1+\check{\alpha}/2}(Q_1)} \leq \check{K},$$

and the partial derivatives of u_1 are the limits of the corresponding partial derivatives of $u_{1\delta}$. Thus, $u_1 \in C(\bar{\Omega}_{\hat{t}}) \cap C^{2,1}(\Omega_{\hat{t}})$.

Next, we assume that $u_n \in C(\bar{\Omega}_{\hat{t}}) \cap C^{2,1}(\Omega_{\hat{t}})$ and show that $u_{n+1} \in C(\bar{\Omega}_{\hat{t}}) \cap C^{2,1}(\Omega_{\hat{t}})$. For $0 < \delta < a$, let $L_\delta u = (x+\delta)^q u_t - u_{xx}$, and we consider the problem,

$$L_\delta u_{(n+1)\delta} = G(u_n(x,t)) \text{ in } \Omega_{\hat{t}},$$

$u_{(n+1)\delta}(x,0) = 0$ on $\bar{D}$, and for $0 < t \leq \hat{t}$,

$$u_{(n+1)\delta}(0,t) = \int_0^a M(x)u_n^p(x,t)dx, u_{(n+1)\delta}(a,t) = \int_0^a N(x)u_n^r(x,t)dx.$$

Since L_δ is an uniformly parabolic operator in $\Omega_{\hat{t}}$, it follows from Theorem 4.2.1 of Ladde, Lakshmikantham and Vatsala [5, pp. 139–142] that the problem has a solution $u_{(n+1)\delta} \in C^{2,1}(\bar{\Omega}_{\hat{t}})$. An argument similar to that in the proof of Theorem 2.4 shows that $u_{(n+1)\delta} \geq 0$, and $u_{(n+1)\delta}$ is nondecreasing with respect to t.

Now,

$$L(\tilde{v} - u_{(n+1)\delta}) = L\tilde{v} - L_\delta u_{(n+1)\delta} + [(x+\delta)^q - x^q](u_{(n+1)\delta})_t \geq 0,$$

$$(\tilde{v} - u_{(n+1)\delta})(x,0) > 0 \text{ on } \bar{D},$$

$$(\tilde{v} - u_{(n+1)\delta})(0,t) = \int_0^a M(x)(\tilde{v}^p(x,t) - u_n^p(x,t))dx \geq 0,\ 0 < t \leq \hat{t},$$

$$(\tilde{v} - u_{(n+1)\delta})(a,t) = \int_0^a N(x)(\tilde{v}^r(x,t) - u_n^r(x,t))dx \geq 0,\ 0 < t \leq \hat{t}.$$

By Theorem 2.2, $\tilde{v} - u_{(n+1)\delta} \geq 0$ in $\Omega_{\hat{t}}$ for any $\delta > 0$.

Furthermore, for any $0 < \delta_1 < \delta_2$, we have

$$\begin{aligned}L_{\delta_2}(u_{(n+1)\delta_1} - u_{(n+1)\delta_2}) &= L_{\delta_1}u_{(n+1)\delta_1} - L_{\delta_2}u_{(n+1)\delta_2} + [(x+\delta_2)^q - (x+\delta_1)^q](u_{(n+1)\delta_1})_t\\ &= [(x+\delta_2)^q - (x+\delta_1)^q](u_{(n+1)\delta_1})_t \geq 0,\end{aligned}$$

$$(u_{(n+1)\delta_1} - u_{(n+1)\delta_2})(x,0) = 0 \text{ on } \bar{D},$$

$$(u_{(n+1)\delta_1} - u_{(n+1)\delta_2})(0,t) = 0 = (u_{(n+1)\delta_1} - u_{(n+1)\delta_2})(a,t),\ 0 < t \leq \hat{t}.$$

By the strong maximum principle (cf. Friedman [4, p. 39]), $u_{(n+1)\delta_1} \geq u_{(n+1)\delta_2}$. Since $\tilde{v}(x,t) \geq u_{(n+1)\delta}(x,t)$, it follows that $\lim_{\delta\to 0} u_{(n+1)\delta}$ exists. Let $\lim_{\delta\to 0} u_{(n+1)\delta}(x,t)$ be denoted by $u_{n+1}(x,t)$.

We are now going to show that $u_{n+1} \in C(\bar{\Omega}_{\hat{t}}) \cap C^{2,1}(\Omega_{\hat{t}})$. For any $(\tilde{x}_1, \tilde{t}_1) \in \Omega_{\hat{t}}$, there is a set $Q_2 = [\tilde{b}_1, \tilde{b}_2] \times [\tilde{t}_2, \tilde{t}_3] \subset \bar{\Omega}_{\hat{t}}$, where $\tilde{b}_1$, $\tilde{b}_2$, $\tilde{t}_2$ and $\tilde{t}_3$ are positive numbers such that $\tilde{b}_1 < \tilde{x}_1 < \tilde{b}_2 < a$ and $\tilde{t}_2 < \tilde{t}_1 \leq \tilde{t}_3$. Since $u_{(n+1)\delta} \leq \tilde{v} < 1$, and $u_n \leq \tilde{v} < 1$, there is some constant $\tilde{p} > 1$ and some positive constants $\tilde{k}_1$, $\tilde{k}_2$ such that

(i) $||u_{(n+1)\delta}||_{L^{\tilde{p}}(Q_2)} \leq ||\tilde{v}||_{L^{\tilde{p}}(Q_2)} \leq \tilde{k}_1$,

(ii) $\|(x+\delta)^{-q} G(u_n)\|_{L^{\tilde{p}}(Q_2)} \leq \tilde{b}_1^{-q} \|G(\tilde{v})\|_{L^{\tilde{p}}(Q_2)} \leq \tilde{k}_2$.

By Ladyženskaja, Solonnikov and Ural′ceva [6, pp. 341–342], $u_{(n+1)\delta} \in W^{2,1}_{\tilde{p}}(Q_2)$. By the embedding theorems there [6, pp. 61 and 80], $W^{2,1}_{\tilde{p}}(Q_2) \hookrightarrow H^{\tilde{\alpha},\tilde{\alpha}/2}(Q_2)$ by choosing $\tilde{p} > 2/(1-\tilde{\alpha})$ with $\tilde{\alpha} \in (0,1)$. Then for some constant $\tilde{k}_3$, $||u_{(n+1)\delta}||_{H^{\tilde{\alpha},\tilde{\alpha}/2}(Q_2)} \leq \tilde{k}_3$. Now,

$$\left\|(x+\delta)^{-q} G(u_n(x,t))\right\|_{H^{\tilde{\alpha},\tilde{\alpha}/2}(Q_2)} \leq \tilde{b}_1^{-q} \|G(\tilde{v})\|_{\infty}$$

$$+ \sup_{\substack{(x_1,t)\in Q_2 \\ (x_2,t)\in Q_2}} \frac{|(x_1+\delta)^{-q} G(u_n(x_1,t)) - (x_2+\delta)^{-q} G(u_n(x_2,t))|}{|x_1 - x_2|^{\tilde{\alpha}}}$$

$$+ \sup_{\substack{(x,t_1)\in Q_2 \\ (x,t_2)\in Q_2}} \frac{(x+\delta)^{-q} |G(u_n(x,t_1)) - G(u_n(x,t_2))|}{|t_1 - t_2|^{\tilde{\alpha}/2}},$$

the first term of which is bounded while the second term satisfies

$$\sup_{\substack{(x_1,t)\in Q_2 \\ (x_2,t)\in Q_2}} \frac{|(x_1+\delta)^{-q} G(u_n(x_1,t)) - (x_2+\delta)^{-q} G(u_n(x_2,t))|}{|x_1 - x_2|^{\tilde{\alpha}}}$$

$$\leq \sup_{\substack{(x_1,t)\in Q_2 \\ (x_2,t)\in Q_2}} \frac{\tilde{b}_1^{-q} |G'(\tilde{v}(\varsigma,t))(u_n(x_1,t) - u_n(x_2,t))|}{|x_1 - x_2|^{\tilde{\alpha}}} \quad \text{for some } \varsigma \in (x_1, x_2)$$

$$\leq \tilde{b}_1^{-q} ||G'(\tilde{v})||_{\infty} \sup_{\substack{(x_1,t)\in Q_2 \\ (x_2,t)\in Q_2}} \frac{|u_n(x_1,t) - u_n(x_2,t)|}{|x_1 - x_2|^{\tilde{\alpha}}}$$

$$\leq \tilde{k}_4 \text{ for some constant } \tilde{k}_4,$$

and the last term

$$\sup_{\substack{(x,t_1)\in Q_2\\(x,t_2)\in Q_2}} \frac{(x+\delta)^{-q}\,|G(u_n(x,t_1))-G(u_n(x,t_2))|}{|t_1-t_2|^{\tilde{\alpha}/2}}$$

$$\leq \tilde{b}_1^{-q}\,||G'(\tilde{v}(x,\theta))||_\infty \sup_{\substack{(x,t_1)\in Q_2\\(x,t_2)\in Q_2}} \frac{|u_n(x,t_1)-u_n(x,t_2)|}{|t_1-t_2|^{\tilde{\alpha}/2}} \text{ for some } \theta\in(t_1,t_2)$$

$$\leq \tilde{k}_5 \text{ for some constant } \tilde{k}_5.$$

Hence, $\|(x+\delta)^{-q}G(u_n(x,t))\|_{H^{\tilde{\alpha},\tilde{\alpha}/2}(Q_2)} \leq \tilde{k}_6$ for some constant $\tilde{k}_6$ which is independent of δ. By Theorem 4.10.1 of Ladyženskaja, Solonnikov and Ural′ceva [6, pp. 351-352], we have

$$\left\|u_{(n+1)\delta}\right\|_{H^{2+\tilde{\alpha},1+\tilde{\alpha}/2}(Q_2)} \leq \tilde{K}$$

for some constant $\tilde{K}$ which is independent of δ. This implies that $u_{(n+1)\delta}$, $(u_{(n+1)\delta})_t$, $(u_{(n+1)\delta})_x$ and $(u_{(n+1)\delta})_{xx}$ are equicontinuous in Q_2. By the Ascoli-Arzela theorem,

$$\|u_{n+1}\|_{H^{2+\tilde{\alpha},1+\tilde{\alpha}/2}(Q_2)} \leq \tilde{K},$$

and the partial derivatives of u_{n+1} are the limits of the corresponding partial derivatives of $u_{(n+1)\delta}$. Thus, $u_{n+1}\in C(\bar{\Omega}_{\hat{t}})\cap C^{2,1}(\Omega_{\hat{t}})$.

Since the sequence $\{u_n(x,t)\}$ is nondecreasing, $\lim_{n\to\infty}u_n(x,t)$ exists in $\Omega_{\hat{t}}$. Let $\lim_{n\to\infty}u_n(x,t)$ be denoted by $u(x,t)$.

For any $(x_1,t_1)\in\Omega_{\hat{t}}$, there is a set $Q=[b_1,b_2]\times[\tau_1,\tau_2]\subset\bar{\Omega}_{\hat{t}}$, where b_1, b_2, τ_1 and τ_2 are positive numbers such that $b_1<x_1<b_2<a$ and $\tau_1<t_1\leq\tau_2$. Since $u_n\leq\tilde{v}$ in Q and $\tilde{v}<1$, we have for some constant $p_1>1$, and some positive constants k_1, k_2,

(i) $||u_n||_{L^{p_1}(Q)} \leq ||\tilde{v}||_{L^{p_1}(Q)} \leq k_1$,

(ii) $\|x^{-q}G(u_n(x,t))\|_{L^{p_1}(Q)} \leq b_1^{-q}\,\|G(\tilde{v})\|_{L^{p_1}(Q)} \leq k_2$.

By Ladyženskaja, Solonnikov and Ural′ceva [6, pp. 341-342], $u_n\in W^{2,1}_{p_1}(Q)$. By the embedding theorems there [6, pp. 61 and 80], $W^{2,1}_{p_1}(Q)\hookrightarrow H^{\alpha,\alpha/2}(Q)$ by choosing $p_1>2/(1-\alpha)$ with $\alpha\in(0,1)$. Then, $||u_n||_{H^{\alpha,\alpha/2}(Q)}\leq k_3$ for some constant k_3. An argument as before gives

$$\|u_n\|_{H^{2+\alpha,1+\alpha/2}(Q)} \leq K$$

for some constant K which is independent of n. This implies that u_n, $(u_n)_t$, $(u_n)_x$ and $(u_n)_{xx}$ are equicontinuous in Q. By the Ascoli-Arzela theorem,

$$\|u\|_{H^{2+\alpha,1+\alpha/2}(Q)} \leq K,$$

and the partial derivatives of u are the limits of the corresponding partial derivatives of u_n. Thus, $u\in C(\bar{\Omega}_{\hat{t}})\cap C^{2,1}(\Omega_{\hat{t}})$. □

Theorem 2.5 gives a local existence of the solution of the problem (1.1)–(1.3). Let $T=\sup\{\hat{t}:$ such that the problem (1.1)–(1.3) has a solution on $\bar{D}\times[0,\hat{t}]\}$. Similar to Theorem 3 of Chan and Liu [3], we obtain $\lim_{t\to T}\max_{\bar{D}}u(x,t)=1$ if $T<\infty$.

3. QUENCHING AND NON-QUENCHING

Let us consider the eigenvalue problem:

$$\varphi''(x) = -\lambda x^q \varphi(x),\ \varphi(0) = 0 = \varphi(a).$$

By the transformation $\varphi(x) = x^{1/2}y(x)$, the above differential equation gives

$$x^2 y'' + xy' + \left(-\frac{1}{4} + \lambda x^{q+2}\right) y = 0.$$

Let $x = z^{2/(q+2)}$. We have

$$z^2 y'' + zy' + \left[-\frac{1}{(q+2)^2} + \frac{4\lambda}{(q+2)^2} z^2\right] y = 0,$$

whose general solution is given by

$$y(z) = AJ_{1/(q+2)}(2\sqrt{\lambda}z/(q+2)) + BJ_{-1/(q+2)}(2\sqrt{\lambda}z/(q+2)),$$

where $J_{1/(q+2)}$ and $J_{-1/(q+2)}$ denote Bessel functions of the first kind of order $1/(q+2)$ and $-1/(q+2)$ respectively. Let μ be the first zero of $J_{1/(q+2)}(2\sqrt{\lambda}a^{(q+2)/2}/(q+2))$. By McLachlan [7, pp. 29, 75], it is positive. From the eigenvalue problem, the (fundamental) eigenfunction corresponding to μ is given by

$$\psi(x) = x^{1/2} J_{1/(q+2)}\left(\frac{2\sqrt{\mu}}{q+2} x^{(q+2)/2}\right),$$

which is positive for $x \in D$. From $\psi(a) = 0$, we see that μa^q decreases when a increases. Let φ denotes the (normalized) fundamental eigenfunction such that $\int_0^a x^q \varphi(x) dx = 1$.

We now give a criterion for quenching in a finite time.

Theorem 3.1. *If $G(u(x,t)) = f(u(x,t))$, and $\mu a^q < f'(0)$, then u quenches in a finite time. If $G(u(x,t)) = \int_0^a h(x,t) f(u(x,t)) dx$, and $\mu a^{q-1} < \underline{h} f(0)$, where $\underline{h} =$ $\inf h(x,t) > 0$, then u quenches in a finite time.*

Proof. Let $w(t) = \int_0^a x^q u\,(x,t)\,\varphi(x) dx$. Then,

$$\begin{aligned} w_t &= \int_0^a x^q u_t \varphi dx \\ &= \int_0^a u_{xx} \varphi dx + \int_0^a G\,(u)\,\varphi dx \\ &\geq -u\,(a,t)\,\varphi'\,(a) + u\,(0,t)\,\varphi'\,(0) - \mu w + a^{-q} \int_0^a G\,(u)\, x^q \varphi dx. \end{aligned}$$

If $G(u(x,t)) = f(u(x,t))$, then it follows from the Jensen inequality that $w_t \geq -\mu w + a^{-q} f(w)$. Since $f'' \geq 0$, we have $f(w) \geq f(0) + f'(0)w$. Hence

$$w_t \geq a^{-q} f(0) + (a^{-q} f'(0) - \mu) w.$$

A direct calculation gives

$$w \geq \frac{f(0)}{f'(0) - \mu a^q} \left[e^{(a^{-q} f'(0) - \mu)t} - 1 \right].$$

Since $w(t) \leq 1$, and $f'(0) - \mu a^q > 0$, there exists some t_0 such that u reaches 1 somewhere in a finite time.

If $G(u(x,t)) = \int_0^a h(x,t) f(u(x,t)) dx$, then

$$\int_0^a G(u(x,t)) x^q \varphi(x) dx \geq a \underline{h} f(0).$$

Hence, $w_t \geq -\mu w + a^{-q+1} \underline{h} f(0)$. By a direct calculation,

$$w \geq \frac{\underline{h} f(0)}{\mu a^{q-1}} \left(1 - e^{-\mu t}\right).$$

Since $\underline{h} f(0) > \mu a^{q-1}$, u reaches 1 somewhere in a finite time. □

Since μa^q decreases when a increases, the theorem implies that the solution quenches in a finite time if a is sufficiently large.

Theorem 3.2. *For a sufficiently small, the solution u exists globally.*

Proof. Let $\rho(x) = x^{1/2} + 1$, and $\xi(t) = \epsilon(e^{-t} + 1)$, where ϵ is a positive number such that $2\epsilon(a^{1/2} + 1) \leq \sigma$ for some fixed $\sigma < 1$. Then, $0 < \rho(x) \xi(t) \leq \sigma < 1$ for $x \in \bar{D}$ and $t > 0$. Let $c = \max\{\max_{\bar{D}} M(x),\ \max_{\bar{D}} N(x)\}$, and a be chosen to satisfy further

$$\epsilon > ca \max\{\sigma^p, \sigma^r\}.$$

Then,

$$\begin{aligned}
\rho(0) \xi(t) &= \epsilon \left(e^{-t} + 1\right) \\
&\geq ca \left(a^{1/2} + 1\right)^p \epsilon^p \left(e^{-t} + 1\right)^p \\
&\geq \left[\epsilon \left(e^{-t} + 1\right)\right]^p \int_0^a M(x) \rho^p(x) \, dx \\
&= \int_0^a M(x) \left(\rho(x) \xi(t)\right)^p dx,
\end{aligned}$$

$$\begin{aligned}
\rho(a) \xi(t) &= \left(a^{1/2} + 1\right) \epsilon \left(e^{-t} + 1\right) \\
&\geq ca \left(a^{1/2} + 1\right)^r \epsilon^r \left(e^{-t} + 1\right)^r \\
&\geq \left[\epsilon \left(e^{-t} + 1\right)\right]^r \int_0^a N(x) \rho^r(x) \, dx \\
&= \int_0^a N(x) \left(\rho(x) \xi(t)\right)^r dx.
\end{aligned}$$

On the other hand,

$$\begin{aligned} L(\rho(x)\xi(t)) - G(\rho(x)\xi(t)) &= -x^q\rho(x)\epsilon e^{-t} + \frac{1}{4}x^{-3/2}\xi(t) - G(\rho(x)\xi(t)) \\ &\geq -\epsilon a^q(a^{1/2}+1) + \frac{1}{4}\epsilon a^{-3/2} - G(2\epsilon(a^{1/2}+1)). \end{aligned}$$

Let us choose a to further satisfy

$$\frac{1}{4}a^{-3/2}\epsilon \geq \epsilon a^q(a^{1/2}+1) + G(2\epsilon(a^{1/2}+1)).$$

Then, $L(\rho(x)\xi(t)) \geq G(\rho(x)\xi(t))$ in Ω. An argument similar to the proof of Theorem 2.4 shows that $\rho(x)\xi(t) \geq u(x,t)$ for $x \in \bar{D}$ and any $t > 0$. Hence, the solution u is bounded above by $\sigma < 1$. This proves the theorem. □

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