

PRODUCT INTEGRATION ON TIME SCALES

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ABSTRACT. We introduce the notion of product Δ -integral of a matrix function defined on an arbitrary time scale, and thus generalize the classical definition of product integral. We prove that every Riemann Δ -integrable matrix function is also product Δ -integrable, and investigate the properties of the indefinite product Δ -integral, including its relation to linear systems of dynamic equations. Finally, we generalize the notion of the matrix exponential function.

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1. INTRODUCTION

The concept of product integration goes back to V. Volterra (see e.g. [9], [10]). Given a matrix function $A : [a, b] \rightarrow \mathbb{R}^{n \times n}$ (where $\mathbb{R}^{n \times n}$ denotes the set of all real $n \times n$ matrices), a partition $a = t_0 < t_1 < \dots < t_m = b$, and a collection of tags $\xi_i \in [t_{i-1}, t_i]$, $i \in \{1, \dots, m\}$, we form the product

$$(I + A(\xi_m)(t_m - t_{m-1})) \cdots (I + A(\xi_1)(t_1 - t_0))$$

(where I stands for the identity matrix). The product integral of A over $[a, b]$, which is classically denoted by the symbol $\prod_a^b (I + A(t)dt)$, is defined as the limit of these products as the norm of the partition approaches zero.

The interest in product integration stems mainly from the fact that if the matrix function A is a Riemann integrable, then the indefinite product integral

$$Y(t) = \prod_a^t (I + A(s)ds), \quad t \in [a, b],$$

is continuous and satisfies the integral equation

$$Y(t) = I + \int_a^t A(s)Y(s)ds.$$

Thus, if A is continuous, then $Y'(t) = A(t)Y(t)$, $Y(a) = I$, and Y is a fundamental matrix of the linear system of equations $y'(t) = A(t)y(t)$, where $y : [a, b] \rightarrow \mathbb{R}^n$.

Product integration has also applications in the theory of stochastic processes, physics, differential equations in the complex domain, etc. (see e.g. [4], [5], [9]). A modern and accessible treatment of product integration, which represents an adaptation of Lebesgue's integration theory, is available in [4].

Inspired by the work of M. Bohner and G. Guseinov on Riemann integration (see [6] and Chapter 5 of [2]), we present a treatment of product integration on time scales. We assume that the reader is familiar with the basic concepts of the calculus on time scales (see e.g. Chapter 1 of [1] or [2]). We start by investigating the class of product Δ -integrable functions and prove that every Riemann Δ -integrable function is also product Δ -integrable. Then we focus on the properties of the indefinite product Δ -integral. The final section describes one of the differences between the classical and time scale theories. As a corollary of the previous results, we obtain the existence-uniqueness theorem for the linear system of dynamic equations $y^\Delta(t) = A(t)y(t)$ (this result was already proved by different methods in [1] or [3]) and a generalized definition of the matrix exponential function in terms of the product Δ -integral.

2. BASIC DEFINITIONS

Let \mathbb{T} be a time scale. We use the symbol $[a, b]_{\mathbb{T}}$ to denote a compact interval in \mathbb{T} , i.e. if $a, b \in \mathbb{T}$, then $[a, b]_{\mathbb{T}} = \{t \in \mathbb{T}; a \leq t \leq b\}$. The open and half-open intervals are defined in an analogous way.

A partition of $[a, b]_{\mathbb{T}}$ is a finite sequence of points

$$\{t_0, t_1, \dots, t_m\} \subset [a, b]_{\mathbb{T}}, \quad a = t_0 < t_1 < \dots < t_m = b.$$

Given such a partition, we put $\Delta t_i = t_i - t_{i-1}$. A tagged partition consists of a partition and a sequence of tags $\{\xi_1, \dots, \xi_m\}$ such that $\xi_i \in [t_{i-1}, t_i)$ for every $i \in \{1, \dots, m\}$. The set of all tagged partitions of $[a, b]_{\mathbb{T}}$ will be denoted by the symbol $D(a, b)$. If not stated otherwise, we will always assume that the division points are called $\{t_0, t_1, \dots, t_m\}$ and the tags $\{\xi_1, \dots, \xi_m\}$.

We say that a tagged partition D' is a refinement of a tagged partition D provided that all division points of D are also division points of D' (the tags in D' might be arbitrary); in this case we write $D' \prec D$.

If $\delta > 0$, then $D_\delta(a, b)$ denotes the set of all tagged partitions of $[a, b]_{\mathbb{T}}$ such that for every $i \in \{1, \dots, m\}$, either $\Delta t_i \leq \delta$, or $\Delta t_i > \delta$ and $\sigma(t_{i-1}) = t_i$. Note that in

Definition 2.1. A bounded function $f : [a, b]_{\mathbb{T}} \rightarrow \mathbb{R}$ is called Riemann Δ -integrable if there exists a number $S \in \mathbb{R}$ with the property that for every $\varepsilon > 0$, there exists a $\delta > 0$ such that $|\sum_{i=1}^m f(\xi_i)\Delta t_i - S| < \varepsilon$ for every $D \in D_\delta(a, b)$. The number S is called the Riemann Δ -integral of f over $[a, b]_{\mathbb{T}}$ and we write

$$\int_a^b f(t)\Delta t = S.$$

The properties of the Riemann Δ -integral are similar to the properties of the classical Riemann integral; we will use the following facts (see [6] or Chapter 5 of [2] for the corresponding proofs):

Theorem 2.2. *If f is Riemann Δ -integrable on $[a, b]_{\mathbb{T}}$, then it is also Riemann Δ -integrable on every subinterval $[c, d]_{\mathbb{T}}$ of $[a, b]_{\mathbb{T}}$.*

Theorem 2.3. *Every rd-continuous function is Riemann Δ -integrable.*

Theorem 2.4. *The product of two Riemann Δ -integrable functions is again Riemann Δ -integrable.*

Theorem 2.5. *If $f : [a, b]_{\mathbb{T}} \rightarrow \mathbb{R}$ is a Riemann Δ -integrable function, then for every $\varepsilon > 0$, there exists a $\delta > 0$ such that $\sum_{i=1}^m (M_i - m_i)\Delta t_i < \varepsilon$ for every $D \in D_\delta(a, b)$, where $M_i = \sup_{u \in [t_{i-1}, t_i]} f(u)$ and $m_i = \inf_{u \in [t_{i-1}, t_i]} f(u)$.*

Theorem 2.6. *Let $f : [a, b]_{\mathbb{T}} \rightarrow \mathbb{R}$ be a rd-continuous function. Then the indefinite integral $F(t) = \int_a^t f(s)\Delta s$ satisfies $F^\Delta(t) = f(t)$ for every $t \in [a, b)$.*

The Riemann Δ -integral of a matrix function $A : [a, b]_{\mathbb{T}} \rightarrow \mathbb{R}^{n \times n}$, where $A = \{a_{ij}\}_{i,j=1}^n$, is defined in a straightforward way as the matrix

$$\left\{ \int_a^b a_{ij}(t)\Delta t \right\}_{i,j=1}^n$$

(provided it exists); we denote it by $\int_a^b A(t)\Delta t$.

To be able to study convergence of matrix functions, we have to introduce a norm on the space $\mathbb{R}^{n \times n}$. Since all norms on a finite-dimensional space are equivalent, it usually doesn't matter which particular norm we choose. However, it is convenient to take a norm that satisfies $\|I\| = 1$, and $\|A \cdot B\| \leq \|A\| \cdot \|B\|$ for every $A, B \in \mathbb{R}^{n \times n}$. These conditions are satisfied e.g. by the operator norm

$$\|A\| = \sup\{\|Ax\|; x \in \mathbb{R}^n, \|x\| \leq 1\},$$

where $\|Ax\|$ and $\|x\|$ denote the Euclidean norms of vectors in \mathbb{R}^n .

We now proceed to the notion of product Δ -integral of a matrix function, a concept which represents a time-scale generalization of the classical product integral.

Given a matrix function $A : [a, b]_{\mathbb{T}} \rightarrow \mathbb{R}^{n \times n}$ and a tagged partition $D \in D(a, b)$, we denote

$$P(A, D) = \prod_{i=m}^1 (I + A(\xi_i)\Delta t_i) = (I + A(\xi_m)\Delta t_m) \cdots (I + A(\xi_1)\Delta t_1).$$

(The order is important because matrix multiplication is in general not commutative.)

Definition 2.7. A bounded matrix function $A : [a, b]_{\mathbb{T}} \rightarrow \mathbb{R}^{n \times n}$ is called product Δ -integrable if there exists a matrix $P \in \mathbb{R}^{n \times n}$ with the property that for every $\varepsilon > 0$, there exists a $\delta > 0$ such that $\|P(A, D) - P\| < \varepsilon$ for every $D \in D_\delta(a, b)$. The matrix P is called the product Δ -integral of A over $[a, b]_{\mathbb{T}}$ and we write

$$\prod_a^b (I + A(t)\Delta t) = P.$$

When $a = b$, we make the agreement that $\prod_a^a (I + A(t)\Delta t) = I$ for every function A .

Theorem 2.8. Let $a, b \in \mathbb{T}$, $a < b$.

- (i) If $\mathbb{T} = \mathbb{R}$, then $A : [a, b]_{\mathbb{T}} \rightarrow \mathbb{R}^{n \times n}$ is product Δ -integrable if and only if it is product integrable in the classical sense, and then

$$\prod_a^b (I + A(t)\Delta t) = \prod_a^b (I + A(t)dt).$$

- (ii) If $h > 0$ and $\mathbb{T} = h\mathbb{Z}$, then every function $A : [a, b]_{\mathbb{T}} \rightarrow \mathbb{R}^{n \times n}$ is product Δ -integrable and

$$\prod_a^b (I + A(t)\Delta t) = \prod_{k=b/h-1}^{a/h} (I + A(kh)h).$$

Proof. The first part is obvious. To prove the second, we note that if $\delta \leq h$, then the only tagged partition $D \in D_\delta(a, b)$ consists of division points $t_0 = a, t_1 = a + h, \dots, t_m = b$ and tags $\xi_i = t_{i-1}$, and therefore

$$P(A, D) = \prod_{i=m}^1 (I + A(\xi_i)\Delta t_i) = \prod_{i=m}^1 (I + A(t_{i-1})h) = \prod_{k=b/h-1}^{a/h} (I + A(kh)h).$$

□

Theorem 2.9. Let $A : [a, b]_{\mathbb{T}} \rightarrow \mathbb{R}^{n \times n}$ and $t \in \mathbb{T}$, $a \leq t \leq \sigma(t) \leq b$. Then

$$\prod_t^{\sigma(t)} (I + A(s)\Delta s) = I + A(t)\mu(t).$$

Proof. The statement is obvious if t is right-dense. Let t be right-scattered and $0 < \delta \leq \mu(t)$. The only tagged partition $D \in D_\delta(t, \sigma(t))$ is $t = t_0 < t_1 = \sigma(t)$, $\xi_1 = t_0$, and therefore $P(A, D) = I + A(t)\mu(t)$. □

3. PRODUCT Δ -INTEGRABLE FUNCTIONS

The main objective of the present section is to prove that every Riemann Δ -integrable function is also product Δ -integrable. The basic idea of the proof is due to L. Schlesinger, who established the result for $\mathbb{T} = \mathbb{R}$ in [8].

Theorem 3.1. *A function $A : [a, b]_{\mathbb{T}} \rightarrow \mathbb{R}^{n \times n}$ is product Δ -integrable if and only if for every $\varepsilon > 0$ there is a $\delta > 0$ such that $\|P(A, D_1) - P(A, D_2)\| < \varepsilon$ for each pair of tagged partitions $D_1, D_2 \in D_{\delta}(a, b)$.*

Proof. The “only if” part is obvious; we prove the “if” part. For every $n \in \mathbb{N}$, we find a $\delta_n > 0$ such that $\|P(A, D_1) - P(A, D_2)\| < 1/n$ for each pair $D_1, D_2 \in D_{\delta_n}(a, b)$. We define

$$X_n = \{P(A, D); D \in D_{\delta_n}(a, b)\}.$$

According to the assumption, each of the sets X_n is bounded. Therefore the intersection $\bigcap_{i=1}^{\infty} \overline{X_n}$ is nonempty, and contains a matrix $P \in \mathbb{R}^{n \times n}$. Given $\varepsilon > 0$, we find $n \in \mathbb{N}$ such that $1/n < \varepsilon$, and put $\delta = \delta_n$. Then every $D \in D_{\delta}(a, b)$ belongs to X_n , and thus $\|P(A, D) - P\| \leq 1/n < \varepsilon$. This proves that $\prod_a^b (I + A(t)\Delta t) = P$. \square

Definition 3.2. Given an interval $I \subseteq \mathbb{T}$ and a matrix function $A : I \rightarrow \mathbb{R}^{n \times n}$, the oscillation of A on I is defined as the number

$$\omega(A, I) = \sup_{u, v \in I} \|A(u) - A(v)\|.$$

Lemma 3.3. *If $A : [a, b]_{\mathbb{T}} \rightarrow \mathbb{R}^{n \times n}$ is Riemann Δ -integrable, then for every $\varepsilon > 0$, there exists a $\delta > 0$ such that $\sum_{i=1}^m \omega(A, [t_{i-1}, t_i])\Delta t_i < \varepsilon$ for every $D \in D_{\delta}(a, b)$.*

Proof. Assume that $A = \{a_{ij}\}_{i,j=1}^n$. For each pair $i, j \in \{1, \dots, n\}$, the function a_{ij} is Riemann Δ -integrable, and thus for every $\varepsilon > 0$, there exists a $\delta > 0$ such that

$$\begin{aligned} & \sum_{i=1}^m \sup_{u, v \in [t_{i-1}, t_i]} |a_{ij}(u) - a_{ij}(v)| \Delta t_i \\ &= \sum_{i=1}^m \left(\sup_{u \in [t_{i-1}, t_i]} a_{ij}(u) - \inf_{v \in [t_{i-1}, t_i]} a_{ij}(v) \right) \Delta t_i < \varepsilon \end{aligned}$$

for every $D \in D_{\delta}(a, b)$ and $i, j \in \{1, \dots, n\}$ (see Theorem 2.5). If $\|A\|_1$ denotes the matrix norm $\sum_{i,j=1}^n |a_{ij}|$ and I is an arbitrary interval, then

$$\sup_{u, v \in I} \|A(u) - A(v)\|_1 = \sup_{u, v \in I} \sum_{i,j=1}^n |a_{ij}(u) - a_{ij}(v)| \leq \sum_{i,j=1}^n \sup_{u, v \in I} |a_{ij}(u) - a_{ij}(v)|.$$

Consequently,

$$\sum_{i=1}^m \sup_{u, v \in [t_{i-1}, t_i]} \|A(u) - A(v)\|_1 \Delta t_i$$

$$\leq \sum_{i=1}^m \sum_{j=1}^n \sup_{u,v \in [t_{i-1}, t_i]} |a_{ij}(u) - a_{ij}(v)| \Delta t_i \leq n^2 \varepsilon.$$

The result follows from the fact that the norms $\|\cdot\|_1$ and $\|\cdot\|$ are equivalent. □

Lemma 3.4. *Let $A : [a, b]_{\mathbb{T}} \rightarrow \mathbb{R}^{n \times n}$ be a matrix function, which satisfies $\|A(t)\| \leq M$ for every $t \in [a, b]_{\mathbb{T}}$. If $\{[u_i, v_i]\}_{i=1}^m$ is a system of disjoint intervals in $[a, b]_{\mathbb{T}}$ and $\xi_i \in [u_i, v_i)$ for every $i \in \{1, \dots, m\}$, then*

$$\left\| \prod_{i=1}^m (I + A(\xi_i)(v_i - u_i)) \right\| \leq e^{M(b-a)}.$$

Proof.

$$\begin{aligned} \left\| \prod_{i=1}^m (I + A(\xi_i)(v_i - u_i)) \right\| &\leq \prod_{i=1}^m \|I + A(\xi_i)(v_i - u_i)\| \\ &\leq \prod_{i=1}^m (1 + M(v_i - u_i)) \leq \prod_{i=1}^m e^{M(v_i - u_i)} \leq e^{M(b-a)}. \end{aligned}$$

□

Lemma 3.5. *Let $A : [a, b]_{\mathbb{T}} \rightarrow \mathbb{R}^{n \times n}$ be a matrix function, which satisfies $\|A(t)\| \leq M$ for every $t \in [a, b]_{\mathbb{T}}$. If $\delta > 0$ and if $D, D' \in D_{\delta}(a, b)$ are such that $D' \prec D$, then*

$$\|P(A, D) - P(A, D')\| \leq e^{M(b-a)} \sum_{\substack{i \in \{1, \dots, m\}, \\ \Delta t_i \leq \delta}} \left(\omega(A, [t_{i-1}, t_i]) \Delta t_i + (M \Delta t_i)^2 e^{M \Delta t_i} \right),$$

where $\{t_0, \dots, t_m\}$ are division points of the partition D . In case the function A is product Δ -integrable, then also

$$\begin{aligned} &\left\| P(A, D) - \prod_a^b (I + A(t) \Delta t) \right\| \\ &\leq e^{M(b-a)} \sum_{\substack{i \in \{1, \dots, m\}, \\ \Delta t_i \leq \delta}} \left(\omega(A, [t_{i-1}, t_i]) \Delta t_i + (M \Delta t_i)^2 e^{M \Delta t_i} \right). \end{aligned}$$

Proof. Every partition D' such that $D' \prec D$ can be obtained from D by a sequence of refinements

$$D = D_0 \rightarrow D_1 \rightarrow D_2 \rightarrow \dots \rightarrow D_m = D',$$

where $D_i \in D(a, b)$ and $D_i \prec D_{i-1}$ for every $i \in \{1, \dots, m\}$, and D_{i-1} and D_i coincide everywhere on $[a, b]_{\mathbb{T}}$ except the interval $[t_{i-1}, t_i)$ (i.e. the i -th refinement is performed only on $[t_{i-1}, t_i)$). We now fix $i \in \{1, \dots, m\}$, and analyze the refinement $D_{i-1} \rightarrow D_i$. It is easy to observe that if $\Delta t_i > \delta$, then $\sigma(t_{i-1}) = t_i$, which means that it is impossible to refine $[t_{i-1}, t_i)$, and $P(A, D_{i-1}) = P(A, D_i)$. Now, suppose that $\Delta t_i \leq \delta$. We write $D_{i-1} = D_I \cup D_{II}^1 \cup D_{III}$ and $D_i = D_I \cup D_{II}^2 \cup D_{III}$, where $D_I \in D(a, t_{i-1})$, $D_{III} \in D(t_i, b)$, and $D_{II}^1, D_{II}^2 \in D(t_{i-1}, t_i)$ are such that $D_{II}^2 \prec D_{II}^1$. We know that D_{II}^1 consists of two division points t_{i-1}, t_i , and a single tag $\xi_i \in [t_{i-1}, t_i)$.

Assume that the division points of D_{II}^2 are $t_{i-1} = u_0 < u_1 < \dots < u_l = t_i$, and that the corresponding tags are $\eta_j \in [u_{j-1}, u_j]$. Consequently,

$$\begin{aligned} \|P(A, D_{i-1}) - P(A, D_i)\| &= \|P(A, D_{III})(P(A, D_{II}^2) - P(A, D_{II}^1))P(A, D_I)\| \\ &\leq \|P(A, D_{III})\| \cdot \left\| \prod_{j=l}^1 (I + A(\eta_j)\Delta u_j) - I - A(\xi_i)\Delta t_i \right\| \cdot \|P(A, D_I)\|. \end{aligned}$$

We know that $\|P(A, D_{III})\| \cdot \|P(A, D_I)\| \leq e^{M(b-a)}$ (see Lemma 3.4), and

$$\begin{aligned} \left\| \prod_{j=l}^1 (I + A(\eta_j)\Delta u_j) - I - A(\xi_i)\Delta t_i \right\| &\leq \left\| \sum_{j=1}^l (A(\eta_j) - A(\xi_i))\Delta u_j \right\| \\ &+ \left\| \sum_{p=2}^l \sum_{1 \leq r_p < \dots < r_1 \leq l} A(\eta_{r_1}) \dots A(\eta_{r_p})\Delta u_{r_1} \dots \Delta u_{r_p} \right\| \\ &\leq \omega(A, [t_{i-1}, t_i])\Delta t_i + \sum_{p=2}^l \sum_{1 \leq r_p < \dots < r_1 \leq l} M^p \Delta u_{r_1} \dots \Delta u_{r_p} \\ &= \omega(A, [t_{i-1}, t_i])\Delta t_i + \prod_{j=1}^l (1 + M\Delta u_j) - 1 - \sum_{j=1}^l M\Delta u_j \\ &\leq \omega(A, [t_{i-1}, t_i])\Delta t_i + e^{M\Delta t_i} - 1 - M\Delta t_i \\ &\leq \omega(A, [t_{i-1}, t_i])\Delta t_i + (M\Delta t_i)^2 e^{M\Delta t_i}. \end{aligned}$$

(The last inequality follows from $e^x - 1 - x \leq x^2 e^x$, which holds for every $x \geq 0$.)
Consequently,

$$\begin{aligned} \|P(A, D) - P(A, D')\| &\leq \sum_{i=1}^m \|P(A, D_{i-1}) - P(A, D_i)\| \\ &= \sum_{\substack{i \in \{1, \dots, m\}, \\ \Delta t_i \leq \delta}} \|P(A, D_{i-1}) - P(A, D_i)\| \\ &\leq e^{M(b-a)} \sum_{\substack{i \in \{1, \dots, m\}, \\ \Delta t_i \leq \delta}} \left(\omega(A, [t_{i-1}, t_i])\Delta t_i + (M\Delta t_i)^2 e^{M\Delta t_i} \right). \end{aligned}$$

The second part of the theorem follows from the fact that if A is product Δ -integrable, then D' can be chosen so that $P(A, D')$ is arbitrarily close to $\prod_a^b (I + A(t)\Delta t)$. \square

Lemma 3.6. *If $A : [a, b]_{\mathbb{T}} \rightarrow \mathbb{R}^{n \times n}$ is Riemann Δ -integrable, then for every $\varepsilon > 0$, there exists a $\delta > 0$ such that if $D, D' \in D_\delta(a, b)$ and $D' \prec D$, then $\|P(A, D) - P(A, D')\| < \varepsilon$.*

Proof. Assume that $\|A(t)\| \leq M$ for every $t \in [a, b]_{\mathbb{T}}$. For a given $\varepsilon > 0$, we find a $\delta > 0$ such that

$$(b - a)\delta M^2 e^{2M(b-a)} < \varepsilon/2$$

and

$$\sum_{i=1}^m \omega(A, [t_{i-1}, t_i]) \Delta t_i < \frac{\varepsilon}{2e^{M(b-a)}}$$

for every $D \in D_\delta(a, b)$ (see Lemma 3.3). Thus, if $D \in D_\delta(a, b)$, then

$$e^{M(b-a)} \sum_{\substack{i \in \{1, \dots, m\}, \\ \Delta t_i \leq \delta}} \omega(A, [t_{i-1}, t_i]) \Delta t_i < \varepsilon/2,$$

$$\begin{aligned} e^{M(b-a)} \sum_{\substack{i \in \{1, \dots, m\}, \\ \Delta t_i \leq \delta}} (M \Delta t_i)^2 e^{M \Delta t_i} &\leq e^{M(b-a)} \delta M^2 e^{M \delta} \sum_{i=1}^m \Delta t_i \\ &\leq \delta M^2 e^{2M(b-a)} (b-a) < \varepsilon/2. \end{aligned}$$

By Lemma 3.5, this means that if $D' \prec D$, then $\|P(A, D) - P(A, D')\| < \varepsilon$. □

Theorem 3.7. *Every Riemann Δ -integrable function is product Δ -integrable.*

Proof. Choose $\varepsilon > 0$. According to Lemma 3.6, there exists a $\delta > 0$ such that $\|P(A, D) - P(A, D')\| < \varepsilon/2$ whenever $D, D' \in D_\delta(a, b)$ and $D' \prec D$. Consider a pair of tagged partitions $D_1, D_2 \in D_\delta(a, b)$. These partitions have a common refinement, i.e. a partition D such that $D \prec D_1, D \prec D_2$ (the tags in D can be chosen arbitrarily). Then

$$\|P(A, D_1) - P(A, D_2)\| \leq \|P(A, D_1) - P(A, D)\| + \|P(A, D) - P(A, D_2)\| < \varepsilon.$$

The statement of the theorem therefore follows from Theorem 3.1. □

Example 3.8. Every constant function A is Riemann Δ -integrable, and therefore also product Δ -integrable. If $\mathbb{T} = h\mathbb{Z}$, then (see Theorem 2.8)

$$\prod_a^b (I + A \Delta t) = (I + Ah)^{(b-a)/h}.$$

Now, let $\mathbb{T} = \mathbb{R}$. For every $m \in \mathbb{N}$, let D_m be a tagged partition that divides $[a, b]_{\mathbb{T}}$ into m subintervals of length $(b-a)/m$ (the tags can be arbitrary). Then

$$\prod_a^b (I + A \Delta t) = \lim_{m \rightarrow \infty} P(A, D_m) = \lim_{m \rightarrow \infty} \left(I + \frac{A(b-a)}{m} \right)^m = e^{A(b-a)}.$$

Theorem 3.9. *If $A : [a, b]_{\mathbb{T}} \rightarrow \mathbb{R}^{n \times n}$ is Riemann Δ -integrable and $c \in [a, b]_{\mathbb{T}}$, then*

$$\prod_a^b (I + A(t) \Delta t) = \prod_c^b (I + A(t) \Delta t) \cdot \prod_a^c (I + A(t) \Delta t).$$

Proof. For every $k \in \mathbb{N}$, put $\delta_k = 1/k$ and choose a pair of tagged partitions $D_k^1 \in D_{\delta_k}(a, c)$, $D_k^2 \in D_{\delta_k}(c, b)$. Letting $D_k = D_k^1 \cup D_k^2$, we obtain a sequence of tagged partitions $\{D_k\}_{k=1}^\infty$ of interval $[a, b]_{\mathbb{T}}$ such that $D_k \in D_{\delta_k}(a, b)$. Note also that $P(A, D_k) = P(A, D_k^2)P(A, D_k^1)$. Consequently,

$$\begin{aligned} \prod_a^b (I + A(t)\Delta t) &= \lim_{k \rightarrow \infty} P(A, D_k) = \lim_{k \rightarrow \infty} P(A, D_k^2) \cdot \lim_{k \rightarrow \infty} P(A, D_k^1) \\ &= \prod_c^b (I + A(t)\Delta t) \cdot \prod_a^c (I + A(t)\Delta t). \end{aligned}$$

□

4. INDEFINITE PRODUCT INTEGRAL

This section focuses on the properties of the indefinite product Δ -integral, i.e. the integral considered as a function of its upper bound. The proofs in the case $\mathbb{T} = \mathbb{R}$ were given by L. Schlesinger (see [8]).

Theorem 4.1. *If $A : [a, b]_{\mathbb{T}} \rightarrow \mathbb{R}^{n \times n}$ is Riemann Δ -integrable, then the indefinite product Δ -integral*

$$Y(t) = \prod_a^t (I + A(s)\Delta s), \quad t \in [a, b]_{\mathbb{T}},$$

is continuous on $[a, b]_{\mathbb{T}}$.

Proof. Let $t_0 \in [a, b)$ be a right-dense point; we prove that Y is right-continuous at t_0 (left-continuity at left-dense points is proved in a similar way). Let $t_0 \leq t_0 + h \leq b$. Assume that $\|A(t)\| \leq M$ for every $t \in [a, b]_{\mathbb{T}}$. Lemma 3.5 applied to interval $[t_0, t_0 + h]_{\mathbb{T}}$ with $\delta = h$ gives

$$\left\| I + A(t_0)h - \prod_{t_0}^{t_0+h} (I + A(s)\Delta s) \right\| \leq e^{Mh}(\omega(A, [t_0, t_0 + h])h + (Mh)^2 e^{Mh}).$$

Since $\omega(A, [t_0, t_0 + h]) \leq 2M$, we obtain

$$\lim_{h \rightarrow 0^+} \prod_{t_0}^{t_0+h} (I + A(s)\Delta s) = I.$$

Therefore (see Theorem 3.9)

$$\lim_{h \rightarrow 0^+} (Y(t_0 + h) - Y(t_0)) = Y(t_0) \left(\prod_{t_0}^{t_0+h} (I + A(s)\Delta s) - I \right) = 0.$$

Theorem 4.2. *If $A : [a, b]_{\mathbb{T}} \rightarrow \mathbb{R}^{n \times n}$ is Riemann Δ -integrable, then the indefinite product Δ -integral*

$$Y(t) = \prod_a^t (I + A(s)\Delta s), \quad t \in [a, b]_{\mathbb{T}},$$

satisfies

$$Y(t) = I + \int_a^t A(s)Y(s)\Delta s, \quad t \in [a, b]_{\mathbb{T}}.$$

Proof. It is sufficient to prove the statement for $t = b$. Assume that $\|A(t)\| \leq M$ for every $t \in [a, b]_{\mathbb{T}}$. For every $\varepsilon > 0$, it is possible to find a $\delta > 0$ such that the following five inequalities hold for every $D \in D_\delta(a, b)$:

$$\|P(A, D) - Y(b)\| < \frac{\varepsilon}{4},$$

$$(b - a)\delta M^2 e^{M(b-a)} < \frac{\varepsilon}{8M(b-a)e^{M(b-a)}},$$

$$\sum_{i=1}^m \omega(A, [t_{k-1}, t_k])\Delta t_k < \frac{\varepsilon}{8M(b-a)e^{M(b-a)}},$$

$$\sum_{i=1}^m \omega(Y, [t_{k-1}, t_k])\Delta t_k < \frac{\varepsilon}{4M},$$

$$\left\| \sum_{i=1}^m A(\xi_i)Y(\xi_i)\Delta t_i - \int_a^b A(s)Y(s)\Delta s \right\| < \frac{\varepsilon}{4}.$$

(Y is continuous, and therefore both Y and AY are Riemann Δ -integrable. The third and the fourth inequality follow from Lemma 3.3.) Take arbitrary $D \in D_\delta(a, b)$ and define

$$Y^k = \prod_{i=k}^1 (I + A(\xi_i)\Delta t_i), \quad k = 0, \dots, m.$$

Note that $Y^0 = I$, $Y^m = P(A, D)$,

$$Y^k - Y^{k-1} = A(\xi_k)Y^{k-1}\Delta t_k, \quad k = 1, \dots, m,$$

$$P(A, D) - I = \sum_{k=1}^m (Y^k - Y^{k-1}) = \sum_{k=1}^m A(\xi_k)Y^{k-1}\Delta t_k.$$

We use this equality in the following estimate:

$$\begin{aligned}
 & \left\| Y(b) - I - \int_a^b A(t)Y(t) \Delta t \right\| \leq \|Y(b) - P(A, D)\| \\
 & + \left\| P(A, D) - I - \int_a^b A(t)Y(t) \Delta t \right\| \\
 & < \frac{\varepsilon}{4} + \left\| \sum_{k=1}^m A(\xi_k)Y^{k-1}\Delta t_k - \int_a^b A(t)Y(t) \Delta t \right\| \\
 & \leq \frac{\varepsilon}{4} + \left\| \sum_{k=1}^m A(\xi_k)(Y^{k-1} - Y(t_{k-1}))\Delta t_k \right\| \\
 & + \left\| \sum_{k=1}^m A(\xi_k)(Y(t_{k-1}) - Y(\xi_k))\Delta t_k \right\| \\
 & + \left\| \sum_{k=1}^m A(\xi_k)Y(\xi_k)\Delta t_k - \int_a^b A(t)Y(t) \Delta t \right\| \\
 & < \frac{\varepsilon}{4} + M \sum_{k=1}^m \|Y^{k-1} - Y(t_{k-1})\| \Delta t_k + M \sum_{k=1}^m \omega(Y, [t_{k-1}, t_k])\Delta t_k + \frac{\varepsilon}{4} \\
 & < \frac{3\varepsilon}{4} + M \sum_{k=1}^m \|Y^{k-1} - Y(t_{k-1})\| \Delta t_k.
 \end{aligned}$$

To obtain an estimate for the remaining sum, we apply Lemma 3.5 to interval $[a, t_{k-1}]_{\mathbb{T}}$, where $k \in \{1, \dots, m\}$:

$$\begin{aligned}
 \|Y^{k-1} - Y(t_{k-1})\| & \leq e^{M(b-a)} \sum_{\substack{j \in \{1, \dots, k\}, \\ \Delta t_j \leq \delta}} (\omega(A, [t_{j-1}, t_j])\Delta t_j + (M\Delta t_j)^2 e^{M\Delta t_j}) \\
 & \leq e^{M(b-a)} \sum_{i=1}^m (\omega(A, [t_{j-1}, t_j])\Delta t_j + e^{M(b-a)}(b-a)\delta M^2) \\
 & < \frac{\varepsilon}{4M(b-a)}.
 \end{aligned}$$

Substituting this inequality in the previous one gives

$$\left\| Y(b) - I - \int_a^b A(t)Y(t) \Delta t \right\| < \varepsilon,$$

which completes the proof. □

Corollary 4.3. *If $A : [a, b]_{\mathbb{T}} \rightarrow \mathbb{R}^{n \times n}$ is rd-continuous, then $Y^\Delta(t) = A(t)Y(t)$ for every $t \in [a, b)$.*

Corollary 4.4. *If $A : [a, b]_{\mathbb{T}} \rightarrow \mathbb{R}^{n \times n}$ is rd-continuous and $y_0 \in \mathbb{R}^n$, then the vector function*

$$y(t) = \prod_a^t (I + A(s)\Delta s)y_0$$

is a solution of the dynamic equation $y^\Delta(t) = A(t)y(t)$ such that $y(a) = y_0$.

5. REGRESSIVE FUNCTIONS

The results that we have obtained up to this point were quite similar to the classical theory. We now turn our attention to the main difference between the classical product integral and its time scale generalization.

It is known that if $\mathbb{T} = \mathbb{R}$ and $A : [a, b]_{\mathbb{T}} \rightarrow \mathbb{R}^{n \times n}$ is a Riemann integrable function, then $\prod_a^b (I + A(t)\Delta t)$ is always a regular matrix. However, this statement is not true for a general time scale. Indeed, if $t \in \mathbb{T}$ is a right-scattered point, then by Theorem 2.9,

$$\prod_t^{\sigma(t)} (I + A(s)\Delta s) = I + A(t)\mu(t),$$

but the right side need not be regular (e.g. if $A(t) = (-1/\mu(t))I$).

Definition 5.1. A function $A : \mathbb{T} \rightarrow \mathbb{R}^{n \times n}$ is called regressive if the matrix $I + A(t)\mu(t)$ is regular for every $t \in \mathbb{T}$.

We have seen that regressivity is a necessary condition for the product Δ -integral to be regular, and we now show that the condition is also sufficient. Note that if $\mathbb{T} = \mathbb{R}$, then every matrix function is regressive; if $\mathbb{T} = \mathbb{Z}$, then a matrix function $A : \mathbb{T} \rightarrow \mathbb{R}^{n \times n}$ is regressive if the matrix $I + A(t)$ is regular for every $t \in \mathbb{T}$, which is equivalent to the condition that for every $t \in \mathbb{T}$, -1 is not an eigenvalue of $A(t)$.

Lemma 5.2. If $A_1, \dots, A_m \in \mathbb{R}^{n \times n}$ are arbitrary matrices, then

$$\left\| \prod_{k=m}^1 (I + A_k) - I \right\| \leq \exp \left(\sum_{k=1}^m \|A_k\| \right) - 1.$$

Proof.

$$\begin{aligned} \left\| \prod_{k=m}^1 (I + A_k) - I \right\| &= \left\| \sum_{j=1}^m \left(\sum_{1 \leq i_j < \dots < i_1 \leq m} A_{i_1} \cdots A_{i_j} \right) \right\| \\ &\leq \sum_{j=1}^m \left(\sum_{1 \leq i_j < \dots < i_1 \leq m} \|A_{i_1}\| \cdots \|A_{i_j}\| \right) \\ &\leq \sum_{j=1}^m \frac{1}{j!} \left(\sum_{i_1, \dots, i_j=1}^m \|A_{i_1}\| \cdots \|A_{i_j}\| \right) \\ &= \sum_{j=1}^m \frac{1}{j!} (\|A_1\| + \cdots + \|A_m\|)^j \leq \exp \left(\sum_{k=1}^m \|A_k\| \right) - 1. \end{aligned}$$

If we now take an arbitrary $A : [a, b]_{\mathbb{T}} \rightarrow \mathbb{R}^{n \times n}$, $D \in D(a, b)$, and put $A_k = A(\xi_k)\Delta t_k$, we obtain

$$\left\| \prod_{k=m}^1 (I + A(\xi_k)\Delta t_k) - I \right\| \leq \exp \left(\sum_{k=1}^m \|A(\xi_k)\| \Delta t_k \right) - 1.$$

Corollary 5.3. *If $A : [a, b]_{\mathbb{T}} \rightarrow \mathbb{R}^{n \times n}$ is Riemann Δ -integrable, then*

$$\left\| \prod_a^b (I + A(t)\Delta t) - I \right\| \leq \exp \left(\int_a^b \|A(t)\| \Delta t \right) - 1.$$

Lemma 5.4. *If $A \in \mathbb{R}^{n \times n}$ satisfies $\|A - I\| < 1$, then A is regular.*

Proof. The condition $\|A - I\| < 1$ implies that the infinite series $\sum_{k=0}^{\infty} (I - A)^k$ is absolutely convergent; let B be the sum of the series. If

$$S_l = \sum_{k=0}^l (I - A)^k,$$

then $S_{l+1} = I + (I - A)S_l = I + S_l(I - A)$. Passing to the limit $l \rightarrow \infty$, we obtain $B = I + (I - A)B = I + B(I - A)$, i.e. $BA = AB = I$. □

The basic idea of the following proof is due to P. R. Masani, who established the result for $\mathbb{T} = \mathbb{R}$ in his paper [7].

Theorem 5.5. *If $A : [a, b]_{\mathbb{T}} \rightarrow \mathbb{R}^{n \times n}$ is a regressive Riemann Δ -integrable function, then $\prod_a^b (I + A(t)\Delta t)$ is a regular matrix.*

Proof. Assume that $\|A(t)\| \leq M$ for every $t \in [a, b]_{\mathbb{T}}$. Choose $\delta > 0$ such that $e^{M\delta} < 2$ and let $D \in D_{\delta}(a, b)$. Then

$$\prod_a^b (I + A(t)\Delta t) = \prod_{i=m}^1 \prod_{t_{i-1}}^{t_i} (I + A(t)\Delta t).$$

Choose $i \in \{1, \dots, m\}$. If $\Delta t_i \leq \delta$, then (see Corollary 5.3)

$$\left\| \prod_{t_{i-1}}^{t_i} (I + A(t)\Delta t) - I \right\| \leq \exp \left(\int_{t_{i-1}}^{t_i} \|A(t)\| \Delta t \right) - 1 \leq e^{M\delta} - 1 < 1,$$

and $\prod_{t_{i-1}}^{t_i} (I + A(t)\Delta t)$ is a regular matrix according to Lemma 5.4. If $\Delta t_i > \delta$, then $\sigma(t_{i-1}) = t_i$, and (see Theorem 2.9) $\prod_{t_{i-1}}^{t_i} (I + A(t)\Delta t) = I + A(t_{i-1})\mu(t_{i-1})$, which is again a regular matrix. Thus the statement follows from the fact that a product of regular matrices is a regular matrix. □

To conclude the present section, we now show that the well-known existence-uniqueness theorem for a linear system of dynamic equations $y^{\Delta}(t) = A(t)y(t)$ (see [1] or [3]) can be obtained as a simple corollary of our previous results.

Definition 5.6. If $a < b$, we define $\prod_b^a(I + A(t)\Delta t) = \left(\prod_a^b(I + A(t)\Delta t)\right)^{-1}$ provided the right-hand side exists.

Theorem 5.7. If $A : \mathbb{T} \rightarrow \mathbb{R}^{n \times n}$ is a regressive rd-continuous function and $t_0 \in \mathbb{T}$, then the function

$$Y(t) = \prod_{t_0}^t (I + A(s)\Delta s), \quad t \in \mathbb{T},$$

represents the unique solution of the dynamic equation $Y^\Delta(t) = A(t)Y(t)$ such that $Y(t_0) = I$.

Proof. Equation $Y(t_0) = I$ is clearly satisfied. We also know that $Y^\Delta(t) = A(t)Y(t)$ holds for $t > t_0$, but we have to prove it for every $t \in \mathbb{T}$. Put

$$Z(t) = \prod_a^t (I + A(s)\Delta s), \quad t \in \mathbb{T}.$$

Then

$$\begin{aligned} Y(t) &= \prod_{t_0}^t (I + A(s)\Delta s) \\ &= \prod_{t_0}^t (I + A(s)\Delta s) \prod_a^{t_0} (I + A(s)\Delta s) \prod_{t_0}^a (I + A(s)\Delta s) = Z(t)C, \end{aligned}$$

where $C = \prod_{t_0}^a (I + A(s)\Delta s)$. Therefore

$$Y^\Delta(t) = (Z(t)C)^\Delta = A(t)Z(t)C = A(t)Y(t), \quad t \in \mathbb{T}^\kappa.$$

To prove uniqueness, consider a matrix function $W : \mathbb{T} \rightarrow \mathbb{R}^{n \times n}$ such that $W^\Delta(t) = A(t)W(t)$ and $W(t_0) = I$. Using the standard rules for calculating Δ -derivatives (see Theorem 5.3 in [1]), we obtain

$$(Y^{-1})^\Delta = -(Y^\sigma)^{-1}Y^\Delta Y^{-1} = -(Y^\sigma)^{-1}A,$$

$$(Y^{-1}W)^\Delta = (Y^{-1})^\sigma W^\Delta + (Y^{-1})^\Delta W = (Y^{-1})^\sigma AW - (Y^\sigma)^{-1}AW = 0.$$

This means that $Y^{-1}W$ is a constant function, and because its value at t_0 is I , we have $Y(t) = W(t)$ for every $t \in \mathbb{T}$. \square

Theorem 5.8. If $A : \mathbb{T} \rightarrow \mathbb{R}^{n \times n}$ is a regressive rd-continuous function, $t_0 \in \mathbb{T}$ and $y_0 \in \mathbb{R}^n$, then the vector function

$$y(t) = \prod_{t_0}^t (I + A(s)\Delta s)y_0$$

represents the unique solution of the dynamic equation $y^\Delta(t) = A(t)y(t)$ such that $y(t_0) = y_0$.

Proof. We only have to prove the uniqueness. Assume to the contrary the existence of solutions $y_1, y_2 : \mathbb{T} \rightarrow \mathbb{R}^n$ such that $y_1(t_0) = y_2(t_0)$, but $y_1(t) \neq y_2(t)$ for some $t \neq t_0$. Then the function $z(t) = y_1(t) - y_2(t)$ satisfies $z^\Delta(t) = A(t)z(t)$, $z(t_0) = 0$, $z(t) \neq 0$. Now, consider the function

$$Z(t) = \prod_{t_0}^t (I + A(s)\Delta s), \quad t \in \mathbb{T}.$$

Let W be a matrix function obtained by adding the vector function z to every column of Z . Then $Z^\Delta(t) = A(t)Z(t)$ and $W^\Delta(t) = A(t)W(t)$ for every $t \in \mathbb{T}^\kappa$, $Z(t_0) = W(t_0)$, but $Z(t) \neq W(t)$, which contradicts Theorem 5.7. \square

Using the method of variation of constants, it is easy to extend the last result to the nonhomogeneous case $y^\Delta(t) = A(t)y(t) + f(t)$. Since the details are given in Chapter 5 of [1], we don't repeat them here.

6. CONCLUSION AND OPEN QUESTIONS

The indefinite product Δ -integral might be viewed as a generalization of the matrix exponential function:

Theorem 6.1. *Let \mathbb{T} be a time scale and $A : \mathbb{T} \rightarrow \mathbb{R}^{n \times n}$ a regressive Riemann Δ -integrable function. Then the function*

$$e_A(t, t_0) = \prod_{t_0}^t (I + A(s)\Delta s), \quad t_0, t \in \mathbb{T},$$

has the following properties:

- (i) $e_0(t, s) \equiv I$ and $e_A(t, t) \equiv I$,
- (ii) $e_A(\sigma(t), s) = (I + A(t)\mu(t))e_A(t, s)$,
- (iii) $e_A(t, s)^{-1} = e_A(s, t)$,
- (iv) $e_A(t, s)e_A(s, r) = e_A(t, r)$.

The theorem follows easily from our previous results. The matrix exponential function corresponding to a rd-continuous regressive function A was introduced in Chapter 5 of [1]; in our approach, we require the function A to be only Riemann Δ -integrable and regressive.

The whole theory of product Δ -integral presented in this paper can be easily modified to obtain the corresponding notion of product ∇ -integral, which provides a solution to the equation $y^\nabla(t) = A(t)y(t)$.

The classical theory of product integration is concerned not only with functions $A : [a, b] \rightarrow \mathbb{R}^{n \times n}$, but more generally with functions $A : [a, b] \rightarrow X$, where X is an arbitrary Banach algebra with a unit element. The corresponding results due to P. R. Masani can be found in [7] and [9]. Some theorems and proofs from our paper

are still valid in this general setting; however, the proof of Lemma 3.3 fails. This leads to the following two questions:

- If X is a Banach algebra and $A : [a, b]_{\mathbb{T}} \rightarrow X$ is a Riemann Δ -integrable (i.e. Graves integrable) function, does it follow that A is product Δ -integrable?
- If the answer to the previous question is affirmative, does the indefinite product Δ -integral Y satisfy the integral equation

$$Y(t) = 1 + \int_a^t A(s)Y(s)\Delta s$$

(where 1 is the unit element of X)?

By expanding the product integral in the Peano-Baker series, which clearly satisfies the above integral equation, Masani obtained positive answers to both questions in the case $\mathbb{T} = \mathbb{R}$. It seems plausible that this method might be generalized to an arbitrary time scale (the Peano-Baker series on time scales was already investigated in the paper [3]).

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