NEW OSCILLATION CRITERIA FOR CERTAIN EVEN ORDER DELAY DIFFERENTIAL EQUATIONS

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ABSTRACT. For even order delay differential equation of the form

$$[p(t)|x^{(n-1)}(t)|^{\alpha-1}x^{(n-1)}(t)]' + F(t, x(g(t)) = 0, \quad n \text{ even}$$

where $p \in C^1([t_0, \infty); (0, \infty))$, $F \in C([t_0, \infty) \times R; R)$, $g \in C([t_0, \infty); R)$, and $\alpha > 0$ is a constant, we obtain several new oscillation criteria without assumptions that has been required for the related results obtained before, our results generalize and improve many known conclusions.

AMS (MOS) Subject Classification. 34D.

1. INTRODUCTION

We are interested in obtaining results on the oscillation behavior of even order delay differential equations of the form

$$[p(t)|x^{(n-1)}(t)|^{\alpha-1}x^{(n-1)}(t)]' + F(t, x(g(t)) = 0, \quad n \text{ even}. \tag{1.1}$$

Where $\alpha > 0$ is a constant.

Throughout this paper, we shall suppose that

(A_1) $p(t) \in C^1([t_0, \infty); (0, \infty)), p'(t) \geq 0, R(t) = \int_{t_0}^{t} \frac{ds}{p^{\frac{1}{\alpha}}(s)} \to \infty (t \to \infty)$.

(A_2) $g(t) \in C([t_0, \infty); R)$. There exists a function $\sigma(t) \in C^1([t_0, \infty); R^+)$ such that $\sigma(t) \leq inf\{t, g(t)\}$, $\lim_{t\to\infty} \sigma(t) = \lim_{t\to\infty} g(t) = \infty$ and $\sigma'(t) > 0$ for $t \geq t_0$.

(A_3) $F(t, x) \in C([t_0, \infty) \times R; R)$, $sgnF(t, x) = sgnx$. There exists a function $q(t) \in C([t_0, \infty); R^+)$ such that $F(t, x)sgnx \geq q(t)|x|^\alpha$ for $x \neq 0$ and $t \geq t_0$;

By a solution of Eq. (1.1), we mean a function $x(t) \in C^1([T_x, \infty); R)$ for some $T_x \geq t_0$ which has the property that

$$p(t)|x^{(n-1)}(t)|^{\alpha-1}x^{(n-1)}(t) \in C^1([T_x, \infty); R)$$

and satisfies Eq. (1.1) on $[T_x, \infty)$. A nontrivial solution of Eq. (1.1) is called oscillatory if it has arbitrary large zero. Otherwise, it is called nonoscillatory. Eq. (1.1) is called oscillatory if all of its solutions are oscillatory.

We say that a function $H = H(t,s)$ belongs to the function class X, if $H \in C(D; R_+)$, where $D = \{(t,s) : t_0 \le s \le t < \infty\}$, which satisfies $H(t,t) = 0, H(t,s) > 0$ for $t > s$, and has partial derivative $\dfrac{\partial H}{\partial s}$ on D such that

$$\frac{\partial H}{\partial s} = -h(t,s)\sqrt{H(t,s)} \ , \tag{1.2}$$

where $h(t,s)$ are locally nonnegative continuous functions on D.

Recently, oscillation of differential equations has become an important area of research due to the fact that such equations arise in many real life problems. In the past few years, there have been some results on the oscillation theory for higher order functional differential equations. We refer the reader to the research papers [1-14] and the references cited there in.

The special case of Eq. (1.1) of the following differential equations

$$x''(t) + q(t)x(t) = 0, \quad t \ge t_0, \tag{1.3}$$

$$[r(t)x'(t)]' + c(t)x(t) = 0, \quad t \ge t_0, \tag{1.4}$$

$$[p(t)|x'(t)|^{\alpha-1}x'(t)]' + q(t)|x(t)|^{\alpha-1}x(t) = 0, \quad t \ge t_0 \tag{1.5}$$

and

$$[|x^{(n-1)}(t)|^{\alpha-1}x^{(n-1)}(t)]' + F(t, x(g(t)) = 0, \quad n \ \ even \tag{1.6}$$

were studied in [10], [11], [12] and [13] respectively. In 1989, Philos [10] proved some oscillation criteria of Eq. (1.3). In [11], Li gave some extensions to the results of Philos [10] for equation (1.4). In 1999, J. V. Manojlovic [12] extend the results of Philos and Li and established oscillation criteria for the equation (1.5). In 2004, Zhiting Xu and Yong Xia [13] established oscillation criteria for the equation (1.6) and give the following result.

Theorem A ([13, Theorem 2.2]) Let the function $H \in X$. If there exists a nondecreasing function $\rho \in C^1([t_0,\infty); R^+)$ such that

(C_1) $0 < \inf_{s \ge t_0} \left\{ \liminf_{t\to\infty} \dfrac{H(t,s)}{H(t,t_0)} \right\} \le \infty,$

(C_2) $\liminf_{t\to\infty} \dfrac{1}{H(t,t_0)} \int_{t_0}^{t} \dfrac{\rho(s)[h(t,s)]^{\alpha+1}}{[H(t,s)\sigma^{n-2}(s)\sigma'(s)]^{\alpha}} ds < \infty,$

if there exists a continuous function $\varphi \in C([t_0,\infty); R)$ such that for all $t > T \ge t_0$,

(C_3) $\limsup_{t\to\infty} \int_{t_0}^{\infty} \sigma^{n-2}(s)\sigma'(s)\rho^{-\frac{1}{\alpha}}(s)[\varphi_+(s)]^{\frac{\alpha+1}{\alpha}} ds = \infty,$

(C_4) $\liminf_{t\to\infty} \frac{1}{H(t,T)} \int_T^t \left[H(t,s)\rho(s)q(s) - \frac{\Theta^{-\alpha}(n,\lambda)}{(\alpha+1)^{(\alpha+1)}} \frac{\rho(s)[h(t,s)]^{\alpha+1}}{[H(t,s)\sigma^{n-2}(s)\sigma'(s)]^{\alpha}} \right] ds \geq \varphi(T)$,

where $\varphi_+(s) = \max\{\varphi(s), 0\}, \Theta(n,\lambda) = \frac{\lambda 2^{2-n}}{(n-2)!} \left[\frac{1}{2} - |\lambda - \frac{1}{2}|\right]^{n-2}$, then Eq. (1.6) is oscillatory.

In this paper, our aim is to study the more general equation (1.1) and establish oscillation criteria which further improve Theorem A. we suggest two different approaches which allow one to remove condition (C_2) in Theorem A. A modified integral averaging technique enables one to simplify essentially the proofs of oscillation criteria.

2. MAIN RESULTS

First, we give the following lemmas for our results.

Lemma 2.1 [14] Let $u(t) \in C^n([t_0,\infty); R^+)$. If $u^{(n)}(t)$ is eventually of one sign for all large t, say $t_1 > t_0$, then there exist a $t_x > t_0$ and an integer $l, 0 \leq l \leq n$, with $n+l$ even for $u^{(n)}(t) \geq 0$ or $n+l$ odd for $u^{(n)}(t) \leq 0$ such that $l > 0$ implies that $u^{(k)}(t) > 0$ for $t > t_x, k = 0, 1, 2, \cdots, l-1$, and $l \leq n-1$ implies that $(-1)^{l+k}u^{(k)}(t) > 0$ for $t > t_x, k = l, l+1, \cdots, n-1$.

Lemma 2.2 [14] If the function $u(t)$ is as in Lemma 2.1 and $u^{(n-1)}(t)u^{(n)}(t) \leq 0$ for $t > t_x$, then there exists a constant $\theta, 0 < \theta < 1$, such that

$$u(t) \geq \frac{\theta}{(n-1)!} t^{n-1} u^{(n-1)}(t) \text{ for all large } t,$$

and

$$u'(\frac{t}{2}) \geq \frac{\theta}{(n-2)!} t^{n-2} u^{(n-1)}(t) \text{ for all large } t.$$

Lemma 2.3 [15] Let X and Y are nonnegative, then

$$X^{\lambda} + (\lambda-1)Y^{\lambda} - \lambda X Y^{\lambda-1} \geq 0, \quad \lambda > 1,$$

where equality holds if and only if $X = Y$.

Theorem 2.1 Let function $H \in X$. If there exists a positive, nondecreasing function $\rho(t) \in C^1([t_0,\infty); (0,\infty))$ such that for some $\beta \geq 1$,

$$\limsup_{t\to\infty} \frac{1}{H(t,t_0)} \int_{t_0}^t \left[H(t,s)\rho(s)q(s) - \frac{\beta^{\alpha} C^{-\alpha}(n,\theta)\rho(s)p(s)h_1^{\alpha+1}(t,s)}{(\alpha+1)^{\alpha+1}(\sigma^{n-2}(s)\sigma'(s)H(t,s))^{\alpha}} \right] ds = \infty, \tag{2.1}$$

where

$$h_1(t,s) = h(t,s)\sqrt{H(t,s)} + H(t,s)\frac{\rho'(s)}{\rho(s)}, \quad C(n,\theta) = \frac{\theta}{2(n-2)!},$$

then Eq. (1.1) is oscillatory.

Proof . Suppose to the contrary that Eq. (1.1) has a nonoscillatory solution $x(t)$, without loss of generality, we assume that $x(t)$ is an eventually positive solution of Eq. (1.1), from (A_2), there exists a number $t_1 \geq t_0$ such that $x(t) > 0, \quad x(g(t)) > 0$ for $t \geq t_1$. Then by Eq. (1.1), we have

$$(p(t)|x^{(n-1)}(t)|^{\alpha-1}x^{(n-1)}(t))' = -F(t, x(g(t))) \leq 0, \quad t \geq t_1. \tag{2.2}$$

Therefore $p(t)|x^{(n-1)}(t)|^{\alpha-1}x^{(n-1)}(t)$ is decreasing and $x^{(n-1)}(t)$ is eventually of one sign. We claim that

$$x^{(n-1)}(t) > 0 \quad \text{for} \quad t \geq t_1.$$

Otherwise, if there exists $\widetilde{t_1} \geq t_1$ such that $x^{(n-1)}(\widetilde{t_1}) < 0$, then for all $t \geq \widetilde{t_1}$,

$$p(t)|x^{(n-1)}(t)|^{\alpha-1}x^{(n-1)}(t) \leq p(t_1)|x^{(n-1)}(t_1)|^{\alpha-1}x^{(n-1)}(t_1) = -C(C > 0), \tag{2.3}$$

then we have $x^{(n-1)}(t) \leq -\dfrac{C}{p^{\frac{1}{\alpha}}(t)}, \quad t \geq \widetilde{t_1}.$

Integrating the above inequality from $\widetilde{t_1}$ to t, we have

$$x^{(n-2)}(t) \leq x^{(n-2)}(\widetilde{t_1}) - C(R(t) - R(\widetilde{t_1})).$$

Letting $t \to \infty$, from (A_1), we get $\lim_{t\to\infty} x^{(n-2)}(t) = -\infty$, which implies $x^{(n-1)}(t)$ and $x^{(n-2)}(t)$ are negative for all large t, from Lemma 2.1, no two consecutive derivatives can be eventually negative, for this would imply that $\lim_{t\to\infty} x(t) = -\infty$, a contradiction. Hence $x^{(n-1)}(t) \geq 0$ for $t \geq t_1$. From Eq. (1.1) and $(A_1), (A_3)$ we have

$$\alpha p(t)(x^{(n-1)}(t))^{\alpha-1}x^{(n)}(t) = [p(t)(x^{(n-1)}(t))^{\alpha}]' - p'(t)(x^{(n-1)}(t))^{\alpha} \leq 0, \quad t \geq t_1,$$

this implies that $x^{(n)}(t) \leq 0, t \geq t_1$. From Lemma 2.1 again (note n is even), we have $x'(t) > 0, \quad t \geq t_1$.

Now from Eq. (1.1) and (A_3) we have

$$[p(t)(x^{(n-1)}(t))^{\alpha}]' \leq -q(t)x^{\alpha}(g(t)) \leq -q(t)x^{\alpha}(\sigma(t)), \quad t \geq t_1. \tag{2.4}$$

By Lemma 2.2, we have

$$x'(\frac{\sigma(t)}{2}) \geq \frac{\theta}{(n-2)!}\sigma^{n-2}(t)x^{(n-1)}(\sigma(t)) \quad t \geq t_1. \tag{2.5}$$

Let

$$w(t) = \rho(t)\frac{p(t)(x^{(n-1)}(t))^{\alpha}}{x^{\alpha}(\frac{\sigma(t)}{2})}, \quad t \geq t_1. \tag{2.6}$$

Then for every $t \geq t_1$, we get

$$w'(t) \leq -\rho(t)q(t) + \frac{\rho'(t)}{\rho(t)}w(t) - \alpha C(n, \theta)\frac{\sigma^{n-2}(t)\sigma'(t)}{(\rho(t)p(t))^{\frac{1}{\alpha}}}w^{\frac{\alpha+1}{\alpha}}(t). \tag{2.7}$$

Multiplying inequality (2.7) by $H(t,s)$ and integrating it with respect s from T to t $(T \geq t_1)$, we have

$$\begin{aligned}
&\int_T^t H(t,s)\rho(s)q(s)ds \\
\leq\ & H(t,T)w(T) + \int_T^t h_1(t,s)w(s)ds \\
-\ & \alpha C(n,\theta)\int_T^t H(t,s)\frac{\sigma^{n-2}(s)\sigma'(s)}{(\rho(s)p(s))^{\frac{1}{\alpha}}}w^{\frac{\alpha+1}{\alpha}}(s)ds \\
=\ & H(t,T)w(T) + \int_T^t \left[h_1(t,s)w(s) - \frac{\alpha}{\beta}C(n,\theta)H(t,s)\frac{\sigma^{n-2}(s)\sigma'(s)}{(\rho(s)p(s))^{\frac{1}{\alpha}}}w^{\frac{\alpha+1}{\alpha}}(s)\right]ds \\
-\ & \frac{\alpha(\beta-1)}{\beta}C(n,\theta)\int_T^t H(t,s)\frac{\sigma^{n-2}(s)\sigma'(s)}{(\rho(s)p(s))^{\frac{1}{\alpha}}}w^{\frac{\alpha+1}{\alpha}}(s)ds.
\end{aligned} \tag{2.8}$$

Taking

$$X = \left(\frac{\alpha}{\beta}C(n,\theta)H(t,s)\sigma^{n-2}(s)\sigma'(s)\right)^{\alpha/(\alpha+1)}\frac{w(s)}{(\rho(s)p(s))^{1/(\alpha+1)}},$$

$$Y = \frac{\alpha^{\alpha/(\alpha+1)}\beta^{\alpha^2/(\alpha+1)}}{(\alpha+1)^\alpha}\frac{(\rho(s)p(s))^{\alpha/(\alpha+1)}h_1^\alpha(t,s)}{(C(n,\theta)\sigma^{n-2}(s)\sigma'(s)H(t,s))^{\alpha^2/(\alpha+1)}}, \quad \lambda = \frac{\alpha+1}{\alpha}.$$

According to the Lemma 2.3, we obtain for $t > s \geq t_1$,

$$\begin{aligned}
h_1(t,s)w(s) \quad & -\tfrac{\alpha}{\beta}C(n,\theta)H(t,s)\frac{\sigma^{n-2}(s)\sigma'(s)}{(\rho(s)p(s))^{\frac{1}{\alpha}}}w^{\frac{\alpha+1}{\alpha}}(s) \\
& \leq \tfrac{\beta^\alpha\rho(s)p(s)h_1^{\alpha+1}(t,s)}{(\alpha+1)^{\alpha+1}(C(n,\theta)\sigma^{n-2}(s)\sigma'(s)H(t,s))^\alpha}.
\end{aligned}$$

Hence, (2.8) implies

$$\begin{aligned}
\int_T^t H(t,s)\rho(s)q(s)ds \leq H(t,T)w(T) + \int_T^t \frac{\beta^\alpha\rho(s)p(s)h_1^{\alpha+1}(t,s)}{(\alpha+1)^{\alpha+1}(C(n,\theta)\sigma^{n-2}(s)\sigma'(s)H(t,s))^\alpha}ds \\
-\frac{\alpha(\beta-1)}{\beta}C(n,\theta)\int_T^t H(t,s)\frac{\sigma^{n-2}(s)\sigma'(s)}{(\rho(s)p(s))^{\frac{1}{\alpha}}}w^{\frac{\alpha+1}{\alpha}}(s)ds.
\end{aligned} \tag{2.9}$$

Using the properties of $H(t,s)$, we conclude from (2.9) that, for all $t \geq T \geq t_1$,

$$\begin{aligned}
&\int_T^t \left[H(t,s)\rho(s)q(s) - \frac{\beta^\alpha\rho(s)p(s)h_1^{\alpha+1}(t,s)}{(\alpha+1)^{\alpha+1}(C(n,\theta)\sigma^{n-2}(s)\sigma'(s)H(t,s))^\alpha}\right]ds \\
\leq\ & H(t,T)w(T) \leq H(t,t_0)w(T),
\end{aligned}$$

and for all $t \geq t_1$,

$$\begin{aligned}
&\int_{t_0}^t \left[H(t,s)\rho(s)q(s) - \frac{\beta^\alpha\rho(s)p(s)h_1^{\alpha+1}(t,s)}{(\alpha+1)^{\alpha+1}(C(n,\theta)\sigma^{n-2}(s)\sigma'(s)H(t,s))^\alpha}\right]ds \\
\leq\ & H(t,t_0)\left[\int_{t_0}^T \rho(s)q(s)ds + w(T)\right],
\end{aligned} \tag{2.10}$$

this gives

$$\limsup_{t\to\infty}\frac{1}{H(t,t_0)}\int_{t_0}^{t}\left[H(t,s)\rho(s)q(s)-\frac{\beta^{\alpha}\rho(s)p(s)h_1^{\alpha+1}(t,s)}{(\alpha+1)^{\alpha+1}(C(n,\theta)\sigma^{n-2}(s)\sigma'(s)H(t,s))^{\alpha}}\right]ds$$
$$\leq\int_{t_0}^{T}\rho(s)q(s)ds+w(T)<+\infty,$$

which contradicts to (2.1). This completes the proof of Theorem 2.1.

Corollary 2.1. Every solution of Eq. (1.1) is oscillatory provided that

$$\limsup_{t\to\infty}\frac{1}{H(t,t_0)}\int_{t_0}^{t}H(t,s)\rho(s)q(s)ds=\infty,$$

and

$$\limsup_{t\to\infty}\frac{1}{H(t,t_0)}\int_{t_0}^{t}\frac{\rho(s)p(s)h_1^{\alpha+1}(t,s)}{(\sigma^{n-2}(s)\sigma'(s)H(t,s))^{\alpha}}ds<\infty,$$

where $H(t,s)$, $h_1(t,s)$, $\rho(t)$ are the same as in Theorem 2.1.

Remark 2.1. If $\beta=1$, Theorem 2.1 reduce to Theorem 1 in [13]. We note that it suffices to satisfy (2.1) in Theorem 2.1 for any $\beta\geq 1$. Parameter β introduce in Theorem 2.1 plays an important role in the results that follow, and it is particularly important in the sequel that $\beta>1$.

With an appropriate choice of the functions H and ρ, one can derive from Theorem 2.1 a number of oscillation criteria for Eq. (2.1). For example, consider a Kamenev-type function $H(t,s)$ defined by $H(t,s)=(t-s)^{\mu}$, $\mu>1$, $(t,s)\in D$, choosing $\rho(t)=t^{\mu}$, then $h_1(t,s)=\dfrac{\mu(t-s)^{\mu-1}t}{s}$. Based on the above results we obtain the following corollary.

Corollary 2.2. Every solution of Eq. (1.1) is oscillatory provided that for some $\beta\geq 1$,

$$\limsup_{t\to\infty}\frac{1}{t^{\mu}}\int_{t_0}^{t}\left[(t-s)^{\mu}s^{\mu}q(s)-\frac{\beta^{\alpha}\mu^{\alpha+1}}{(\alpha+1)^{\alpha+1}}\frac{p(s)t^{\alpha+1}s^{\mu-\alpha-1}(t-s)^{\mu-\alpha-1}}{(C(n,\theta)\sigma^{n-2}(s)\sigma'(s))^{\alpha}}\right]ds=\infty.$$

Corollary 2.3. Suppose that

$$\limsup_{t\to\infty}\frac{1}{t^{\mu}}\int_{t_0}^{t}s^{\mu}(t-s)^{\mu}q(s)ds=\infty,$$

and

$$\limsup_{t\to\infty}\frac{1}{t^{\mu-\alpha-1}}\int_{t_0}^{t}\frac{p(s)s^{\mu-\alpha-1}(t-s)^{\mu-\alpha-1}}{(\sigma^{n-2}(s)\sigma'(s))^{\alpha}}ds<\infty$$

hold, then every solution of Eq. (1.1) is oscillatory.

Example. Consider the following equation

$$[t^{-\nu}|x^{n-1}(t)|^{\alpha-1}x^{n-1}(t)]'+q(t)|x\left(\frac{t}{2}\right)|^{\alpha-1}x\left(\frac{t}{2}\right)=0, \tag{2.11}$$

$t \geq t_0$, where ν is arbitrary positive constant and $\alpha > 0$.

Here, $p(t) = t^{-\nu}, \quad q(t) \in C[t_0, \infty)$. It follows from Theorem 41 in [15] that

$$(t-s)^{\mu} \geq t^{\mu} - \mu s t^{\mu-1}, \quad t \geq s \geq t_0.$$

If $q(s) \geq \dfrac{c}{s^{\mu+1}}, c > 0$, we have

$$\limsup_{t\to\infty} \frac{1}{t^{\mu}} \int_{t_0}^{t} s^{\mu}(t-s)^{\mu} q(s)ds \geq \limsup_{t\to\infty} \frac{1}{t^{\mu}} \int_{t_0}^{t} s^{\mu}(t^{\mu} - \mu s t^{\mu-1})q(s)ds = \infty,$$

let $\mu = \alpha + 1, \quad$ if $\quad \nu + (n-2)\alpha > 1$, we have

$$\limsup_{t\to\infty} \frac{1}{t^{\mu-\alpha-1}} \int_{t_0}^{t} \frac{p(s)s^{\mu-\alpha-1}(t-s)^{\mu-\alpha-1}}{(\sigma^{n-2}(s)\sigma'(s))^{\alpha}} ds = 2^{(n-1)\alpha} \limsup_{t\to\infty} \int_{t_0}^{t} \frac{1}{s^{\nu+(n-2)\alpha}} ds < \infty,$$

then from Corollary 2.3, Eq. (2.11) is oscillatory.

Theorem 2.2. Let the functions H and ρ be the same as in Theorem 2.1, and assume also that

$$\inf_{s\geq t_0} \left\{ \liminf_{t\to\infty} \frac{H(t,s)}{H(t,t_0)} \right\} > 0. \tag{2.12}$$

If there exists a function $\varphi \in C([t_0,\infty), R)$ such that for all $T \geq t_0$ and for some $\beta > 1$,

$$\limsup_{t\to\infty} \frac{1}{H(t,T)} \int_{T}^{t} \left[H(t,s)\rho(s)q(s) - \frac{\beta^{\alpha} C^{-\alpha}(n,\theta)\rho(s)p(s)h_1^{\alpha+1}(t,s)}{(\alpha+1)^{\alpha+1}(\sigma^{n-2}(s)\sigma'(s)H(t,s))^{\alpha}} \right] ds \geq \varphi(T) \tag{2.13}$$

and

$$\limsup_{t\to\infty} \int_{t_0}^{\infty} \frac{\sigma^{n-2}(s)\sigma'(s)}{(\rho(s)p(s))^{\frac{1}{\alpha}}} \varphi_{+}^{\frac{\alpha+1}{\alpha}}(s)ds = \infty, \tag{2.14}$$

where $\varphi_{+}(s) = \max\{\varphi(s), 0\}$, then every solution of Eq. (1.1) is oscillatory.

Proof. Proceeding as in the proof of Theorem 2.1, introduce the function $w(t)$ through (2.6), we arrive at the inequality (2.9) holds for all $t > T \geq t_1$ and for any $\beta > 1$. Thus we have,

$$\begin{aligned} &\limsup_{t\to\infty} \frac{1}{H(t,T)} \int_{T}^{t} \left[H(t,s)\rho(s)q(s) - \frac{\beta^{\alpha} C^{-\alpha}(n,\theta)\rho(s)p(s)h_1^{\alpha+1}(t,s)}{(\alpha+1)^{\alpha+1}(\sigma^{n-2}(s)\sigma'(s)H(t,s))^{\alpha}} \right] ds \\ \leq \quad & w(T) - \frac{\alpha(\beta-1)}{\beta} C(n,\theta) \liminf_{t\to\infty} \frac{1}{H(t,T)} \int_{T}^{t} H(t,s) \frac{\sigma^{n-2}(s)\sigma'(s)}{(\rho(s)p(s))^{\frac{1}{\alpha}}} w^{\frac{\alpha+1}{\alpha}}(s)ds. \end{aligned} \tag{2.15}$$

It follows from (2.13) that

$$w(T) \geq \varphi(T) + \frac{\alpha(\beta-1)}{\beta} C(n,\theta) \liminf_{t\to\infty} \frac{1}{H(t,T)} \int_{T}^{t} H(t,s) \frac{\sigma^{n-2}(s)\sigma'(s)}{(\rho(s)p(s))^{\frac{1}{\alpha}}} w^{\frac{\alpha+1}{\alpha}}(s)ds \tag{2.16}$$

for all $T \geq t_1$ and for some $\beta > 1$. Consequently,

$$\varphi(T) \leq w(T), \tag{2.17}$$

and

$$\liminf_{t\to\infty} \frac{1}{H(t,t_1)} \int_{t_1}^{t} H(t,s) \frac{\sigma^{n-2}(s)\sigma'(s)}{(\rho(s)p(s))^{\frac{1}{\alpha}}} w^{\frac{\alpha+1}{\alpha}}(s)ds \leq \frac{\beta C^{-1}(n,\theta)}{\alpha(\beta-1)}(w(t_1)-\varphi(t_1)) < \infty. \tag{2.18}$$

Now we shall prove that

$$\int_{t_1}^{\infty} \frac{\sigma^{n-2}(s)\sigma'(s)}{(\rho(s)p(s))^{\frac{1}{\alpha}}} w^{\frac{\alpha+1}{\alpha}}(s)ds < \infty. \tag{2.19}$$

Suppose to the contrary, that is

$$\int_{t_1}^{\infty} \frac{\sigma^{n-2}(s)\sigma'(s)}{(\rho(s)p(s))^{\frac{1}{\alpha}}} w^{\frac{\alpha+1}{\alpha}}(s)ds = \infty. \tag{2.20}$$

Condition (2.12) implies existence of a $\lambda > 0$ such that

$$\liminf_{t\to\infty} \frac{H(t,s)}{H(t,t_0)} > \lambda > 0, \tag{2.21}$$

and there exists a $T_1 > t_1 > t_0$ such that

$$\frac{H(t,T_1)}{H(t,t_0)} \geq \lambda$$

for all $t \geq T_1$.

On the other hand, by virtue of (2.20), for any positive number K, there exists a $T_2 > T_1$ such that for all $t \geq T_2$,

$$\int_{t_1}^{t} \frac{\sigma^{n-2}(s)\sigma'(s)}{(\rho(s)p(s))^{\frac{1}{\alpha}}} w^{\frac{\alpha+1}{\alpha}}(s)ds \geq \frac{K}{\lambda}.$$

Using integration by parts, we conclude that, for all $t \geq T_2$,

$$\begin{aligned}
&\frac{1}{H(t,t_1)} \int_{t_1}^{t} H(t,s) \frac{\sigma^{n-2}(s)\sigma'(s)}{(\rho(s)p(s))^{\frac{1}{\alpha}}} w^{\frac{\alpha+1}{\alpha}}(s)ds \\
&= \frac{1}{H(t,t_1)} \int_{t_1}^{t} \left[-\frac{\partial H(t,s)}{\partial s} \int_{t_1}^{s} \frac{\sigma^{n-2}(\tau)\sigma'(\tau)}{(\rho(\tau)p(\tau))^{\frac{1}{\alpha}}} w^{\frac{\alpha+1}{\alpha}}(\tau)d\tau \right] ds \\
&\geq \frac{K}{\lambda} \frac{1}{H(t,t_0)} \int_{T_1}^{t} \left[-\frac{\partial H(t,s)}{\partial s} \right] ds = \frac{K}{\lambda} \frac{H(t,T_1)}{H(t,t_0)},
\end{aligned} \tag{2.22}$$

it follows from (2.22) that, for all $t \geq T_2$,

$$\frac{1}{H(t,t_1)} \int_{t_1}^{t} H(t,s) \frac{\sigma^{n-2}(s)\sigma'(s)}{(\rho(s)p(s))^{\frac{1}{\alpha}}} w^{\frac{\alpha+1}{\alpha}}(s)ds \geq K.$$

Since K is an arbitrary positive constant,

$$\liminf_{t\to\infty} \frac{1}{H(t,t_1)} \int_{t_1}^{t} H(t,s) \frac{\sigma^{n-2}(s)\sigma'(s)}{(\rho(s)p(s))^{\frac{1}{\alpha}}} w^{\frac{\alpha+1}{\alpha}}(s)ds = \infty.$$

Which contradicts (2.18). Consequently, (2.19) holds, and, by virtue of (2.17),

$$\int_{t_1}^{\infty} \frac{\sigma^{n-2}(s)\sigma'(s)}{(\rho(s)p(s))^{\frac{1}{\alpha}}}\varphi_+^{\frac{\alpha+1}{\alpha}}(s)ds \leq \int_{t_1}^{\infty} \frac{\sigma^{n-2}(s)\sigma'(s)}{(\rho(s)p(s))^{\frac{1}{\alpha}}}w^{\frac{\alpha+1}{\alpha}}(s)ds < \infty.$$

Which contradicts (2.14), therefore, Eq. (1.1) is oscillatory.

Choosing $H(t,s) = (t-s)^{\mu}, \rho(s) = s^{\mu}$, it is not difficult to see that condition (2.12) holds. Consequently, one immediately derives from Theorem 2.2 the following useful oscillation test for Eq .(1.1).

Corollary 2.4. If there exists a function $\varphi \in C([t_0,\infty), R)$ such that for all $T \geq t_0$, and for some $\beta > 1$,

$$\limsup_{t\to\infty} \frac{1}{t^{\mu}} \int_T^t \left[(t-s)^{\mu}s^{\mu}q(s) - \frac{\beta^{\alpha}\mu^{\alpha+1}}{(\alpha+1)^{\alpha+1}} \frac{p(s)t^{\alpha+1}s^{\mu-\alpha-1}(t-s)^{\mu-\alpha-1}}{(C(n,\theta)\sigma^{n-2}(s)\sigma'(s))^{\alpha}}\right] ds \geq \varphi(T) \tag{2.23}$$

and

$$\limsup_{t\to\infty} \int_{t_0}^t \frac{\sigma^{n-2}(s)\sigma'(s)}{p^{\frac{1}{\alpha}}(s)s^{\frac{\mu}{\alpha}}}\varphi_+^{\frac{\alpha+1}{\alpha}}(s)ds = \infty \tag{2.24}$$

hold, then Eq. (1.1) is oscillatory.

Theorem 2.3. Let the functions H, ρ and $\varphi(s)$ be the same as in Theorem 2.2, and assume also that (2.12) be satisfied. If for some $\beta > 1$, and for all $T \geq t_0$,

$$\liminf_{t\to\infty} \frac{1}{H(t,T)} \int_T^t \left[H(t,s)\rho(s)q(s) - \frac{\beta^{\alpha}C^{-\alpha}(n,\theta)\rho(s)p(s)h_1^{\alpha+1}(t,s)}{(\alpha+1)^{\alpha+1}(\sigma^{n-2}(s)\sigma'(s)H(t,s))^{\alpha}}\right] ds \geq \varphi(T) \tag{2.25}$$

and (2.14) holds, then every solution of Eq. (1.1) is oscillatory.

Proof. The conclusion of the theorem follows immediately from the properties of the limits

$$\begin{aligned}\varphi(T) &\leq \liminf_{t\to\infty} \frac{1}{H(t,T)} \int_T^t \left[H(t,s)\rho(s)q(s) - \frac{\beta^{\alpha}C^{-\alpha}(n,\theta)\rho(s)p(s)h_1^{\alpha+1}(t,s)}{(\alpha+1)^{\alpha+1}(\sigma^{n-2}(s)\sigma'(s)H(t,s))^{\alpha}}\right] ds \\ &\leq \limsup_{t\to\infty} \frac{1}{H(t,T)} \int_T^t \left[H(t,s)\rho(s)q(s) - \frac{\beta^{\alpha}C^{-\alpha}(n,\theta)\rho(s)p(s)h_1^{\alpha+1}(t,s)}{(\alpha+1)^{\alpha+1}(\sigma^{n-2}(s)\sigma'(s)H(t,s))^{\alpha}}\right] ds\end{aligned}$$

and Theorem 2.2.

Corollary 2.5. If there exists a function $\varphi \in C([t_0,\infty), R)$ such that for all $T \geq t_0$, and for some $\beta > 1$,

$$\liminf_{t\to\infty} \frac{1}{t^{\mu}} \int_T^t \left[(t-s)^{\mu}s^{\mu}q(s) - \frac{\beta^{\alpha}\mu^{\alpha+1}}{(\alpha+1)^{\alpha+1}} \frac{p(s)t^{\alpha+1}s^{\mu-\alpha-1}(t-s)^{\mu-\alpha-1}}{(C(n,\theta)\sigma^{n-2}(s)\sigma'(s))^{\alpha}}\right] ds \geq \varphi(T) \tag{2.26}$$

and (2.24) hold, then Eq. (1.1) is oscillatory.

If we choosing $H(t,s)=(t-s)^\mu, \rho(s)=1$, then condition (2.12) holds, and we derives from Theorem 2.2 the following oscillation test for Eq. (1.1).

Corollary 2.6. If there exists a function $\varphi \in C([t_0,\infty),R)$ such that for all $T \geq t_0$, for some $\beta > 1$,

$$\limsup_{t\to\infty} \frac{1}{t^\mu}\int_T^t \left[(t-s)^\mu q(s) - \frac{\beta^\alpha \mu^{\alpha+1} C^{-\alpha}(n,\theta)}{(\alpha+1)^{\alpha+1}} \frac{p(s)(t-s)^{\mu-\alpha-1}}{(\sigma^{n-2}(s)\sigma'(s))^\alpha}\right] ds \geq \varphi(T) \tag{2.27}$$

and

$$\limsup_{t\to\infty} \int_{t_0}^t \frac{\sigma^{n-2}(s)\sigma'(s)\varphi_+^{\frac{\alpha+1}{\alpha}}(s)}{p^{\frac{1}{\alpha}}(s)} ds = \infty \tag{2.28}$$

hold, then Eq. (1.1) is oscillatory.

Remark 2.2. The parameter β in Theorem 2.2 and Theorem 2.3 is strictly larger than one. This allows us to eliminate the conditions similar to (C_2) which have been assumed in most papers on the subject and shorten significantly the proofs of Theorem 2.2 and Theorem 2.3.

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