KAMENEV-TYPE AND INTERVAL OSCILLATION THEOREMS FOR SECOND ORDER NONLINEAR DELAY DYNAMIC EQUATIONS

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ABSTRACT. By means of the generalized Riccati technique, we establish Kamenev-type and interval oscillation theorems for the second-order nonlinear delay dynamic equation

$$(r(t)x^{\Delta}(t))^{\Delta} + p(t)f(x(\tau(t))) = 0$$

on an unbounded time scale $\mathbb{T}$. Our results are extensions of those for second order ordinary differential equations and provide new oscillation criteria for second order delay difference and q-difference equations. Some examples are given to illustrate the significance of our main theorems.

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1. INTRODUCTION

In this paper, we consider the second order nonlinear delay dynamic equation

$$(r(t)x^{\Delta}(t))^{\Delta} + p(t)f(x(\tau(t))) = 0 \tag{1.1}$$

on an unbounded time scale $\mathbb{T}$. In Eq. (1.1), we assume that $r, p \in \mathbb{C}_{rd}(\mathbb{T}, \mathbb{R}^+)$,$\tau \in \mathbb{C}_{rd}(\mathbb{T}, \mathbb{T})$, $\tau(t) \le t$, $\lim_{t\to\infty} \tau(t) = \infty$, $uf(u) > 0$ and $|f(u)| \ge L|u|$ for $u \neq 0$. Define $[t_0, \infty)_{\mathbb{T}}$ by $[t_0, \infty)_{\mathbb{T}} := [t_0, \infty) \cap \mathbb{T}$, and suppose that the functions r and f are sufficiently smooth to ensure that every solution $x(t)$ of Eq. (1.1) that under consideration is continuable to the right and is nontrivial, i.e., $x(t)$ exists on some half-line $[T_x, \infty)$ and satisfies $\sup\{|x(t)| : t \ge T_x\} > 0$ for any $T_x \ge t_0$. A solution x of Eq. (1.1) is said to have a generalized zero at $t^* \in \mathbb{T}$ if $x(t^*)x(\sigma(t^*)) \le 0$. A function x is an oscillatory solution of Eq. (1.1) if and only if x is a solution of Eq. (1.1) that is neither eventually positive nor negative, otherwise it is nonoscillatory. Equation (1.1) is said to be oscillatory if all solutions are oscillatory. Throughout this paper, a knowledge and understanding of time scales and time-scale notation is assumed; for an excellent introduction to the calculus on time scales, see [1, 3, 4].

Equation (1.1) in its general form includes different types of delay differential and difference equation depending on the choice of the time scale $\mathbb{T}$. For example, when $\mathbb{T}=\mathbb{R}$, then Eq. (1.1) becomes the delay differential equation

$$(r(t)x'(t))' + p(t)f(x(\tau(t))) = 0. \tag{1.2}$$

If $\mathbb{T}=\mathbb{N}$, then Eq. (1.1) becomes the delay difference equation

$$\Delta(r_n\Delta x_n) + p_n f(x_{\tau_n}) = 0, \tag{1.3}$$

where $\Delta x_n = x_{n+1} - x_n$. If $\mathbb{T} = q^{\mathbb{N}_0} := \{t : t = q^n, n \in \mathbb{N}_0, q > 1\}$, then Eq. (1.1) becomes the q-difference equation

$$\Delta_q(r(t)\Delta_q x(t)) + p(t)f(x(\tau(t))) = 0, \tag{1.4}$$

where $\Delta_q x(t) = (x(qt) - x(t))/((q-1)t)$.

In recent years, there has been an increasing interest in obtaining sufficient conditions for oscillation of solutions for different classes of second order dynamic equations and delay dynamic equations. We refer to the recent papers [2–9, 12, 13, 15, 16, 17], among those, for oscillation of second order delay dynamic equations, we would like to mention that Erbe, Peterson, Saker's results [9] are more general because the generalized Riccati technique was used. However, the conditions in [9] for oscillation is a kind of requirement that the coefficient functions r, p must require information on the whole set $[t_0,\infty)_{\mathbb{T}}$. This has to be considered a disadvantage in applications, which must be relaxed.

Very recently, Medico and Kong [12, 13] established Kamenev-type and interval oscillation criteria for the self-adjoint second order dynamic equation

$$(r(t)x^{\Delta}(t))^{\Delta} + p(t)x(\sigma(t)) = 0. \tag{1.5}$$

It is clear that the results given in [12, 13] cannot be applied to Eq. (1.1). To develop the qualitative theory of delay dynamic equations on time scales, in this paper, we intend to use a generalized Riccati technique, following the ideas in Erbe, Peterson, Saker [9], Kong [11] and Medico and Kong [12], to obtain several new Kamenev-type oscillation criteria as well as interval criteria for Eq. (1.1). Under the restriction that $p(t) \geq 0$ for Eq. (1.5), one can easily see that our results extend the main results in [12] for Eq. (1.5) to Eq. (1.1). We will apply our results to the delay discrete cases to get some oscillation criteria for delay discrete equation (1.3) and q-difference equation (1.4). Finally, some examples are given to illustrate the significance of our main results.

2. KAMENEV-TYPE CRITERIA

In this section, we employ the generalized Riccati substitution and establish Kamenev-type oscillation criteria for Eq. (1.1).

Following Philos [14], let $D = \{(t,s) \in \mathbb{T}^2 : t \geq s \geq t_0 > 0\}$. For any function $H(t,s) : \mathbb{T}^2 \to \mathbb{R}$, denote by H_1^Δ and H_2^Δ the partial derivatives of H with respect to t and s, respectively. For $\mathbb{E} \subset \mathbb{R}$, denote by $L_{loc}(\mathbb{E})$ the space of functions which are integrable on any compact subset of E. Define

$$\begin{aligned}\mathcal{H}^* =& \big\{H(t,s) \in \mathbb{C}^1(D,\mathbb{R}_+) : (H_2^\Delta(t,\cdot))^2/H(t,\cdot) \in L_{loc}([0,\rho(t)] \cap \mathbb{T}),\\ & H(t,t) = 0,\, H(t,s) \geq 0 \quad \text{and} \quad H_2^\Delta(t,s) \leq 0 \text{ for } t \geq s \geq t_0\big\},\end{aligned}$$

and

$$\begin{aligned}\mathcal{H}_* =& \big\{H(t,s) \in \mathbb{C}^1(D,\mathbb{R}_+) : (H_1^\Delta(\cdot,s))^2/H(\cdot,s) \in L_{loc}([\sigma(s),\infty) \cap \mathbb{T}),\\ & H(t,t) = 0,\, H(t,s) \geq 0 \quad \text{and} \quad H_1^\Delta(t,s) \geq 0 \text{ for } t \geq s \geq t_0\big\}.\end{aligned}$$

We now start with the following Lemma whose proof can be found in [9]

Lemma 2.1. *Assume that*

$$\int_{t_0}^{\infty} \frac{\Delta(t)}{r(t)} = \infty, \tag{2.1}$$

and

$$\int_{t_0}^{\infty} \tau(t)p(t)\Delta t = \infty, \tag{2.2}$$

and assume that Eq. (1.1) *has a positive solution x on $[t_0,\infty) \cap \mathbb{T}$. Then there exists a $T \in [t_0,\infty) \cap \mathbb{T}$ sufficiently large such that*

(1) *$x^\Delta(t) > 0$, $x(t) > tx^\Delta(t)$ for $[T,\infty) \cap \mathbb{T}$;*

(2) *$x(t)$ is strictly increasing and $x(t)/t$ is strictly decreasing on $[T,\infty) \cap \mathbb{T}$.*

The first theorem gives oscillation conditions using functions in $\mathcal{H}^*$.

Theorem 2.2. *Let* (2.1) *and* (2.2) *hold. Assume that there exist functions $H \in \mathcal{H}^*$, $a \in \mathbb{C}_{rd}([t_0,\infty)_\mathbb{T},\mathbb{R})$ and $\delta \in \mathbb{C}_{rd}^1([t_0,\infty)_\mathbb{T},\mathbb{R}^+)$ such that for sufficiently large T,*

$$\begin{aligned}\limsup_{t\to\infty} \frac{1}{H(t,T)}\Big[&\int_T^t H(t,\sigma(s))\delta^\sigma(s)\psi(s)\Delta s\\ &-\frac{1}{4}\int_T^{\rho(t)} H(t,\sigma(s))\phi_1^2(t,s)g(s)\Delta s - H_2^\Delta(t,\rho(t))\eta(\rho(t))\chi_{t-\rho(t)}\Big] = \infty,\end{aligned} \tag{2.3}$$

where

$$g(s) := \frac{\sigma(s)r(s)\delta^2(s)}{s\delta^\sigma(s)}, \quad \eta(s) := \mu(s)\delta(s)a(s)r(s),$$

$$\psi(s) := \frac{Lp(s)\tau(s)}{\sigma(s)} - [a(s)r(s)]^\Delta + \frac{sr(s)a^2(s)}{\sigma(s)},$$

and

$$\phi_1(t,s) := \frac{\delta^\sigma(s)}{\delta(s)}\Big(\frac{\delta^\Delta(s)}{\delta^\sigma(s)} + \frac{2sa(s)}{\sigma(s)}\Big) + \frac{H_2^\Delta(t,s)}{H(t,\sigma(s))}, \quad \chi_t := \begin{cases} 0, & t = 0,\\ 1, & t \in (0,\infty).\end{cases}$$

Then Eq. (1.1) *is oscillatory.*

Proof. Suppose to the contrary that $x(t)$ is a nonoscillatory solution of Eq. (1.1). Then there is a $t_1 \in [t_0,\infty)\cap\mathbb{T}$ such that $x(\tau(t)) \neq 0$ on $[t_1,\infty)\cap\mathbb{T}$. We will only consider the case where $x(\tau(t)) > 0$ for $t \in (t_1,\infty)\cap\mathbb{T}$ as the proof in other case is similar. In view of Lemma 2.1, there is some $t_2 \geq t_1$ such that

$$x^{\Delta}(t) > 0, \quad (r(t)x^{\Delta}(t)) < 0 \quad \text{for } t \geq t_2.$$

Define the generalized Riccati substitution $w(t)$ by

$$w(t) = \delta(t)\Big[\frac{r(t)x^{\Delta}(t)}{x(t)} + r(t)a(t)\Big] \text{ for } t \geq t_2. \tag{2.4}$$

Hence,

$$\begin{aligned} w^{\Delta} &= \frac{\delta^{\Delta}}{\delta}w + \delta^{\sigma}(ra)^{\Delta} + \delta^{\sigma}\,\frac{x(rx^{\Delta})^{\Delta} - r(x^{\Delta})^2}{xx^{\sigma}} \\ &\leq -\delta^{\sigma}\frac{pf\circ x\circ\tau}{x^{\sigma}} - \delta^{\sigma}r\frac{x}{x^{\sigma}}(\frac{x^{\Delta}}{x})^2 + \frac{\delta^{\Delta}}{\delta}w + \delta^{\sigma}(ra)^{\Delta} \end{aligned} \tag{2.5}$$

From the definition of $w(t)$, we see that

$$\Big(\frac{x^{\Delta}}{x}\Big)^2 = \Big(\frac{w}{r\delta} - a\Big)^2 = \Big(\frac{w}{r\delta}\Big)^2 - 2\frac{wa}{r\delta} + a^2. \tag{2.6}$$

Also from Lemma 2.1, since $x(t)/t$ is strictly decreasing, we have

$$\frac{x(\tau(t))}{x^{\sigma}(t)} \geq \frac{\tau(t)}{\sigma(t)} \quad \text{and} \quad \frac{x(t)}{x^{\sigma}(t)} \geq \frac{t}{\sigma(t)}.$$

Substituting the above into (2.5), and noting that (2.6), we obtain

$$\begin{aligned} w^{\Delta}(t) &\leq -\delta^{\sigma}(t)\frac{p(t)f(x(\tau(t)))}{x^{\sigma}(t)} + \delta^{\sigma}(t)(r(t)a(t))^{\Delta} + \frac{\delta^{\Delta}(t)}{\delta}w(t) \\ &\quad - \frac{t\delta^{\sigma}(t)r(t)}{\sigma(t)}\Big[\Big(\frac{w(t)}{r(t)\delta(t)}\Big)^2 - \frac{2a(t)w(t)}{r(t)\delta(t)} + a^2(t)\Big] \\ &\leq -\delta^{\sigma}(t)\psi(t) + \frac{\delta^{\sigma}(t)}{\delta(t)}\Big(\frac{\delta^{\Delta}(t)}{\delta^{\sigma}(t)} + \frac{2ta(t)}{\sigma(t)}\Big)w(t) - \frac{t\delta^{\sigma}(t)}{\sigma(t)r(t)\delta^2(t)}w^2(t) \\ &= -\delta^{\sigma}(t)\psi(t) + \frac{\delta^{\sigma}(t)}{\delta(t)}\Big(\frac{\delta^{\Delta}(t)}{\delta^{\sigma}(t)} + \frac{2ta(t)}{\sigma(t)}\Big)w(t) - \frac{1}{g(t)}w^2(t). \end{aligned} \tag{2.7}$$

Multiplying (2.7), where t is replaced by s, by $H(t,\sigma(s))$, and integrating it with respect s, we get

$$\begin{aligned} \int_{t_2}^{t} H(t,\sigma(s))\delta^{\sigma}(s)\psi(s)\Delta s &\leq \int_{t_2}^{t} H(t,\sigma(s))\frac{\delta^{\sigma}(s)}{\delta(s)}\Big(\frac{\delta^{\Delta}(s)}{\delta^{\sigma}(s)} + \frac{2sa(s)}{\sigma(s)}\Big)w(s)\Delta s \\ &\quad - \int_{t_2}^{t} H(t,\sigma(s))w^{\Delta}(s)\Delta s - \int_{t_2}^{t}\frac{1}{g(s)}H(t,\sigma(s))w^2(s)\Delta s. \end{aligned} \tag{2.8}$$

Integrating by parts and using the fact $H(t,t) = 0$, we get

$$\int_{t_2}^{t} H(t,\sigma(s))w^{\Delta}(s)\Delta s = -H(t,t_2)w(t_2) - \int_{t_2}^{t} H_2^{\Delta}(t,s)w(s)\Delta s. \tag{2.9}$$

Substituting (2.9) into (2.8), we have

$$\int_{t_2}^{t} H(t,\sigma(s))\delta^\sigma(s)\psi(s)\Delta s \le H(t,t_2)w(t_2) + \int_{t_2}^{\rho(t)} H(t,\sigma(s))\phi_1(t,s)w(s)\Delta s$$

$$+\int_{\rho(t)}^{t} H_2^\Delta(t,s)w(s)\Delta s - \int_{t_2}^{\rho(t)} \frac{1}{g(s)}H(t,\sigma(s))w^2(s)\Delta s. \tag{2.10}$$

Noting that $H_2^\Delta(t,s) \le 0$ on D, $H(t,t)=0$ and $w(t) \ge \delta(t)r(t)a(t)$, we have

$$\int_{\rho(t)}^{t} H_2^\Delta(t,s)w(s)\Delta s \le H_2^\Delta(t,\rho(t))w(\rho(t))\mu(\rho(t))\chi_{t-\rho(t)}$$

$$\le H_2^\Delta(t,\rho(t))\eta(\rho(t))\chi_{t-\rho(t)}. \tag{2.11}$$

Combining (2.11) and (2.10), and after completing the square, we get

$$\int_{t_2}^{t} H(t,\sigma(s))\delta^\sigma(s)\psi(s)\Delta s$$

$$\le H(t,t_2)w(t_2) - \int_{t_2}^{\rho(t)} \frac{1}{g(s)}H(t,\sigma(s))\big[w(s)-\frac{1}{2}g(s)\phi_1(t,s)\big]^2\Delta s$$

$$+ H_2^\Delta(t,\rho(t))\eta(\rho(t))\chi_{t-\rho(t)} + \frac{1}{4}\int_{t_2}^{\rho(t)} H(t,\sigma(s))\phi_1^2(t,s)g(s)\Delta s.$$

$$\le H(t,t_2)w(t_2) + \frac{1}{4}\int_{t_2}^{\rho(t)} H(t,\sigma(s))\phi_1^2(t,s)g(s)\Delta s$$

$$+ H_2^\Delta(t,\rho(t))\eta(\rho(t))\chi_{t-\rho(t)} \tag{2.12}$$

Hence,

$$\frac{1}{H(t,t_2)}\Big[\int_{t_2}^{t} H(t,\sigma(s))\delta^\sigma(s)\psi(s)\Delta s - \frac{1}{4}\int_{t_2}^{\rho(t)} H(t,\sigma(s))\phi_1^2(t,s)g(s)\Delta s$$

$$-H_2^\Delta(t,\rho(t))\eta(\rho(t))\chi_{t-\rho(t)}\Big] \le w(t_2) < \infty,$$

which contradicts (2.3). This completes the proof. □

In the sequel, we define

$$\mathbb{T}_1 = \{s\in\mathbb{T} : s \text{ is right-dense}\} \quad \text{and} \quad \mathbb{T}_2 = \{s\in\mathbb{T} : s \text{ is right-scattered}\}. \tag{2.13}$$

The following corollary is from Theorem 2.2 where $H(t,s)=(t-s)^m$, $m>1$.

Corollary 2.3. *For $t\in\mathbb{T}$, let $\mathbb{T}_1(t)=[0,t)\cap\mathbb{T}_1$ and $\mathbb{T}_2(t)=[0,t)\cap\mathbb{T}_2$, and let (2.1) and (2.2) hold. Assume that there exist a constant $m>1$ and a function $\delta\in\mathbb{C}_{rd}^1([t_0,\infty)_\mathbb{T},\mathbb{R}^+)$ such that for sufficiently large T,*

$$\limsup_{t\to\infty}\frac{1}{t^m}\Big[\int_{T}^{t}(t-\sigma(s))^m\delta^\sigma(s)\psi(s)\Delta s - (t-\rho(t))^m g_1(t)$$

$$-\frac{m^2}{4}\int_{\mathbb{T}_1(\rho(t))}(t-s)^{m-2}r(s)\delta(s)ds - \frac{m^2}{4}\sum_{\mathbb{T}_2(\rho(t))}\frac{(t-s)^{2m-2}}{(t-\sigma(s))^m}g_2(t)\Big] = \infty, \tag{2.14}$$

where

$$g_1(t) = \frac{\delta(\rho(t))r(\rho(t))\sigma(\rho(t))\delta^{\Delta}(\rho(t))}{2\rho(t)\delta^{\sigma}(\rho(t))}\chi_{t-\rho(t)} \quad and \quad g_2(t) = \frac{r(t)\sigma(t)\delta^2(t)\mu(t)}{t\delta^{\sigma}(t)}.$$

Then Eq. (1.1) *is oscillatory.*

Proof. Let

$$H(t,s) = (t-s)^m \quad \text{and} \quad a(t) = -\frac{\sigma(t)\delta^{\Delta}(t)}{2t\delta^{\sigma}(t)}.$$

Then, $H \in \mathcal{H}^*$, and

$$H_2^{\Delta}(t,s) = \begin{cases} -m(t-s)^{m-1}, & s \in \mathbb{T}_1, \\ \big[(t-\sigma(s))^m - (t-s)^m\big]/\mu(s), & s \in \mathbb{T}_2. \end{cases}$$

From the mean value theorem, for $t \in \mathbb{T}_2$, there exists $\xi(s) \in [s, \sigma(s)]$ such that

$$0 \geq H_2^{\Delta}(t,s) = -m(t-\xi(s))^{m-1} \geq -m(t-s)^{m-1}. \tag{2.15}$$

Hence, for $t \in \mathbb{T}$ in both cases, we have

$$H_2^{\Delta}(t,\rho(t))\mu(\rho(t))\chi_{t-\rho(t)} = -(t-\rho(t))^m.$$

Substituting this into (2.3), we have

$$\begin{aligned} &\limsup_{t\to\infty} \frac{1}{(t-t_1)^m}\Big[\int_{t_1}^{t} (t-\sigma(s))^m \delta^{\sigma}(s)\psi(s)\Delta s - (t-\rho(t))^m g_1(t) \\ &- \frac{m^2}{4}\int_{\mathbb{T}_1(\rho(t))} (t-s)^{m-2} r(s)\delta(s)ds - \frac{m^2}{4}\sum_{\mathbb{T}_2(\rho(t))} \frac{m^2(t-s)^{2m-2}}{(t-\sigma(s))^m} g_2(t)\Big] = \infty. \end{aligned} \tag{2.16}$$

Notice that, for any $t_1 \in \mathbb{T}$, (2.16) is equivalent to (2.14), and the conclusion holds. □

The next theorem gives oscillation conditions using functions in H_*. Note that this result does not apply to the case where all points in $\mathbb{T}$ are right-dense.

Theorem 2.4. *Let $H \in \mathcal{H}_*$, $\mathbb{T}_1$, $\mathbb{T}_2$ be defined by* (2.13), *and let* (2.1) *and* (2.2) *hold. Assume that there exist functions $H \in \mathcal{H}^*$, $a \in \mathbb{C}_{rd}([t_0,\infty)_{\mathbb{T}}, \mathbb{R}^+)$ and $\delta \in \mathbb{C}^1_{rd}([t_0,\infty)_{\mathbb{T}}, \mathbb{R}^+)$. Then* Eq. (1.1) *is oscillatory provided there exists $\{t_n\}_{n=1}^{\infty} \subset \mathbb{T}_2$, $t_n \to \infty$, such that for sufficient large $t_\alpha \in \mathbb{T}$, one of the following holds:*

(i) $\lim_{n\to\infty} H(t_n, t_\alpha)a(t_n)r(t_n)\delta(t_n) = \infty$, *and*

$$\begin{aligned} &\limsup_{n\to\infty} \frac{1}{H(t_n,t_\alpha)a(t_n)r(t_n)\delta(t_n)}\Big[\int_{t_\alpha}^{t_n} H(\sigma(s),t_\alpha)\delta^{\sigma}(s)\psi(s)\Delta s \\ &-\frac{1}{4}\int_{\sigma(t_\alpha)}^{t_n} H(\sigma(s),t_\alpha)g(s)\phi_2^2(t,s)\Delta s\Big] = \infty; \end{aligned} \tag{2.17}$$

(ii) $\limsup_{n\to\infty} H(t_n,t_\alpha)a(t_n)r(t_n)\delta(t_n)=\infty$, *and*

$$\lim_{n\to\infty}\frac{1}{H(t_n,t_\alpha)a(t_n)r(t_n)\delta(t_n)}\Big[\int_{t_\alpha}^{t_n}H(\sigma(s),t_\alpha)\delta^\sigma(s)\psi(s)\Delta s$$
$$-\frac{1}{4}\int_{\sigma(t_\alpha)}^{t_n}H(\sigma(s),t_\alpha)g(s)\phi_2^2(t,s)\Delta s\Big]=\infty; \tag{2.18}$$

(iii) $\lim_{n\to\infty} H(t_n,t_\alpha)a(t_n)r(t_n)\delta(t_n)<\infty$, *and*

$$\limsup_{n\to\infty}\Big[\int_{t_\alpha}^{t_n}H(\sigma(s),t_\alpha)\delta^\sigma(s)\psi(s)\Delta s-\frac{1}{4}\int_{\sigma(t_\alpha)}^{t_n}H(\sigma(s),t_\alpha)\phi_2^2(t,s)g(s)\Delta s\Big]=\infty, \tag{2.19}$$

where

$$\phi_2(t,s)=\frac{\delta^\sigma(t)}{\delta(t)}\left(\frac{\delta^\Delta(t)}{\delta^\sigma(t)}+\frac{2ta(t)}{\sigma(t)}\right)+\frac{H_1^\Delta(t,s)}{H(\sigma(t),s)},$$

and $\psi(t),g(t)$ *are the same as Theorem* 2.2.

Proof. Assume Eq. (1.1) is not oscillatory. Without loss of generality we may assume that there exists $t_\alpha\in[t_0,\infty)\cap\mathbb{T}$ such that $x(t)>0$ for $t\in[t_\alpha,\infty)\cap\mathbb{T}$. Following the proof of Theorem 2.1, we get (2.7) holds. Multiplying (2.7), where t is replaced by s, by $H(\sigma(s),t_\alpha)$, integrating it with respect to s from t_α to t, and then using the integration by parts formula, and noting that

$$\int_{t_\alpha}^{t}H(\sigma(s),t_\alpha)w^\Delta\Delta s=H(t,t_\alpha)w(t)-\int_{t_\alpha}^{t}H_1^\Delta(s,t_\alpha)w(s)\Delta s,$$

we obtain

$$\int_{t_\alpha}^{t}H(\sigma(s),t_\alpha)\delta^\sigma(s)\psi(s)\Delta s\le -H(t,t_\alpha)w(t)+\left(\int_{t_\alpha}^{\sigma(t_\alpha)}+\int_{\sigma(t_\alpha)}^{t}\right)\Big[H_1^\Delta(s,t_\alpha)w(s)$$
$$+\frac{\delta^\sigma(s)}{\delta(s)}\left(\frac{\delta^\Delta(s)}{\delta^\sigma(s)}+\frac{2sa(s)}{\sigma(s)}\right)H(\sigma(s),t_\alpha)w(s)-\frac{1}{g(s)}H(\sigma(s),t_\alpha)w^2(s)\Big]\Delta s. \tag{2.20}$$

Let

$$R(s,t_\alpha)=H_1^\Delta(s,t_\alpha)w(s)+\frac{\delta^\sigma(s)}{\delta(s)}\left(\frac{\delta^\Delta(s)}{\delta^\sigma(s)}+\frac{2sa(s)}{\sigma(s)}\right)H(\sigma(s),t_\alpha)w(s)$$
$$-\frac{1}{g(s)}H(\sigma(s),t_\alpha)w^2(s).$$

Since $H(t_\alpha,t_\alpha)=0$, while $\sigma(t_\alpha)>t_\alpha$, after completing the square,

$$\int_{t_\alpha}^{\sigma(t_\alpha)}R(s,t_\alpha)\Delta s\le\frac{1}{4}g(t_\alpha)\phi_2^2(t_\alpha,t_\alpha)\mu(t_\alpha)H(\sigma(t_\alpha),t_\alpha),$$

when $\sigma(t_\alpha)=t_\alpha$,

$$\int_{t_\alpha}^{\sigma(t_\alpha)}R(s,t_\alpha)\Delta s=0.$$

So in both cases, we have

$$\int_{t_\alpha}^{\sigma(t_\alpha)} R(s,t_\alpha)\Delta s \le \eta_2(t_\alpha), \tag{2.21}$$

where

$$\eta_2(t) = \begin{cases} 0, & \sigma(t)=t, \\ \frac{1}{4}g(t)\phi_2^2(t,t)\mu(t)H(\sigma(t),t), & \sigma(t)>t, \end{cases} \tag{2.22}$$

and

$$\int_{\sigma(t_\alpha)}^{t} R(s,t_\alpha)\Delta s \le \int_{\sigma(t_\alpha)}^{t} \frac{1}{4}g(s)H(\sigma(s),t_\alpha)\phi_2^2(t,s)\Delta s. \tag{2.23}$$

Substituting (2.21) and (2.23) into (2.20), we have

$$\begin{aligned} &\int_{t_\alpha}^{t} H(\sigma(s),t_\alpha)\delta^\sigma(s)\psi(s)\Delta s \\ &\le -H(t,t_\alpha)w(t)+\eta_2(t_\alpha)+\frac{1}{4}\int_{\sigma(t_\alpha)}^{t} g(s)H(\sigma(s),t_\alpha)\phi_2^2(t,s)\Delta s. \end{aligned} \tag{2.24}$$

Let $t=t_n$ in (2.24), we have

$$\begin{aligned} &\frac{1}{H(t_n,t_\alpha)a(t_n)r(t_n)\delta(t_n)}\Bigg[\int_{t_\alpha}^{t_n} H(\sigma(s),t_\alpha)\delta^\sigma(s)\psi(s)\Delta s \\ &\quad -\frac{1}{4}\int_{\sigma(t_\alpha)}^{t_n} g(s)H(\sigma(s),t_\alpha)\phi_2^2(t,s)\Delta s\Bigg] \le -1+\frac{\eta_2(t_\alpha)}{H(t_n,t_\alpha)a(t_n)r(t_n)\delta(t_n)}. \end{aligned}$$

Taking the lim sup as $n\to\infty$, we have

$$\begin{aligned} \limsup_{n\to\infty} &\frac{1}{H(t_n,t_\alpha)a(t_n)r(t_n)\delta(t_n)}\Bigg[\int_{t_\alpha}^{t_n} H(\sigma(s),t_\alpha)\delta^\sigma(s)\psi(s)\Delta s \\ &-\frac{1}{4}\int_{\sigma(t_\alpha)}^{t_n} g(s)H(\sigma(s),t_\alpha))\phi_2^2(t,s)\Delta s\Bigg] < \infty, \end{aligned}$$

which contradicts (2.17). This completes the proof of (i).

The conclusion with conditions (ii) and (iii) can be obtained easily. We omit the details. □

Corollary 2.5. *Let* $\mathbb{T}_1$, $\mathbb{T}_2$ *be defined by* (2.13), *and let* (2.1) *and* (2.2) *hold. Assume that there exist functions* $H\in\mathcal{H}^*$, $a\in\mathbb{C}_{rd}([t_0,\infty)_\mathbb{T},\mathbb{R}^+)$ *and* $\delta\in\mathbb{C}^1_{rd}([t_0,\infty)_\mathbb{T},\mathbb{R}^+)$ *with* $\delta^\Delta(t)\ge 0, t\in[t_0,\infty)_\mathbb{T}$. *Then* Eq. (1.1) *is oscillatory provided there exists* $\{t_n\}_{n=1}^\infty\subset\mathbb{T}_2$, $t_n\to\infty$, *such that for sufficient large* $t_\alpha\in\mathbb{T}$ *and a constant* $m>1$, *one of the following holds:*

(i) $\lim_{n\to\infty}(t_n)^m a(t_n)r(t_n)\delta(t_n)=\infty,$ *and*

$$\limsup_{n\to\infty}\frac{1}{(t_n)^m a(t_n)r(t_n)\delta(t_n)}\Big[\int_{\sigma(t_\alpha)}^{t_n}(\sigma(s)-t_\alpha)^m\delta^\sigma(s)\psi(s)\Delta s - \frac{1}{4}\int_{\mathbb{T}_1(\sigma(t_\alpha),t_n)}(s-t_\alpha)^m\varphi_1^2(s,t_\alpha)r(s)\delta(s)\Delta s - \frac{1}{4}\sum_{\mathbb{T}_2(\sigma(t_\alpha),t_n)}(\sigma(s)-t_\alpha)^m\varphi_2^2(s,t_\alpha)\frac{\sigma(s)r(s)\delta^2(s)}{s\delta^\sigma(s)}\mu(s)\Big]=\infty; \tag{2.25}$$

(ii) $\limsup_{n\to\infty}(t_n)^m a(t_n)r(t_n)\delta(t_n)=\infty,$ *and*

$$\lim_{n\to\infty}\frac{1}{(t_n)^m a(t_n)r(t_n)\delta(t_n)}\Big[\int_{\sigma(t_\alpha)}^{t_n}(\sigma(s)-t_\alpha)^m\delta^\sigma(s)\psi(s)\Delta s - \frac{1}{4}\int_{\mathbb{T}_1(\sigma(t_\alpha),t_n)}(s-t_\alpha)^m\varphi_1^2(s,t_\alpha)r(s)\delta(s)\Delta s - \frac{1}{4}\sum_{\mathbb{T}_2(\sigma(t_\alpha),t_n)}(\sigma(s)-t_\alpha)^m\varphi_2^2(s,t_\alpha)\frac{\sigma(s)r(s)\delta^2(s)}{s\delta^\sigma(s)}\mu(s)\Big]=\infty; \tag{2.26}$$

(iii) $\lim_{n\to\infty}(t_n)^m a(t_n)r(t_n)\delta(t_n)<\infty,$ *and*

$$\limsup_{n\to\infty}\frac{1}{(t_n)^m a(t_n)r(t_n)\delta(t_n)}\Big[\int_{\sigma(t_\alpha)}^{t_n}(\sigma(s)-t_\alpha)^m\delta^\sigma(s)\psi(s)\Delta s - \frac{1}{4}\int_{\mathbb{T}_1(\sigma(t_\alpha),t_n)}(s-t_\alpha)^m\varphi_1^2(s,t_\alpha)r(s)\delta(s)\Delta s - \frac{1}{4}\sum_{\mathbb{T}_2(\sigma(t_\alpha),t_n)}(\sigma(s)-t_\alpha)^m\varphi_2^2(s,t_\alpha)\frac{\sigma(s)r(s)\delta^2(s)}{s\delta^\sigma(s)}\mu(s)\Big]=\infty, \tag{2.27}$$

where

$$\varphi_1(t,s)=\frac{\delta^\Delta(t)}{\delta^\sigma(t)}+2a(t)+\frac{m}{t-s}\ \textit{and}\ \varphi_2(t,s)=\frac{\delta^\sigma(t)}{\delta(t)}\Big(\frac{\delta^\Delta(t)}{\delta^\sigma(t)}+\frac{2ta(t)}{\sigma(t)}\Big)+\frac{m}{(\sigma(t)-s)}.$$

Proof. Let $H(t,s)=(t-s)^m$. Then $H\in\mathcal{H}^*$, and

$$H_1^\Delta(s,t_\alpha)=\begin{cases} m(s-t_\alpha)^{m-1}, & s\in\mathbb{T}_1,\\ (\sigma(s)-t_\alpha)^m-(s-t_\alpha)^m)/\mu(s), & s\in\mathbb{T}_2.\end{cases}$$

Note from the mean value theorem that for $t\in\mathbb{T}_2$ there exists $\xi(s)\in[s,\sigma(s)]$ such that

$$0\le H_1^\Delta(s,t_\alpha)=m(\xi(s)-t_\alpha)^{m-1}\le m(\sigma(s)-t_\alpha)^{m-1}.$$

Therefore, the conclusion follows from Theorem 2.4. □

3. INTERVAL CRITERIA

Now, we establish analogues of the interval criteria for oscillation of Eq. (1.5) in [12] to the dynamic Eq. (1.1). Further conditions for oscillation of the Kamenev-type are derived from them.

Lemma 3.1. *Let $H \in \mathcal{H}^*$, $\delta \in \mathbb{C}^1_{rd}([t_0,\infty)_{\mathbb{T}}, \mathbb{R}^+)$. Assume* (2.1) *and* (2.2) *hold and $x(t)$ is a solution of* Eq. (1.1) *such that $x(t) > 0$ on a nonempty interval $[c, b) \cap \mathbb{T}$ with $c, b \in \mathbb{T}$. Let $w(t)$ be defined by* (2.4)*. Then*

$$\int_c^b H(b,\sigma(s))\delta^\sigma(s)\psi(s)\Delta s \le H(b,c)w(c) + H_2^\Delta(b,\rho(b))\eta(\rho(b))\chi_{\mu(\rho(b))} + \frac{1}{4}\int_c^{\rho(b)} g(s)H(b,\sigma(s))\phi_1^2(b,s)\Delta s, \tag{3.1}$$

where $\psi(s)$ and $g(s)$ are the same as in Theorem 2.1*.*

Proof. As in the proof of Theorem 2.1 we obtain inequality (2.12). Then (3.1) follows directly from (2.12) with $t_2 = c$ and $t = b$. □

Lemma 3.2. *Let $H \in \mathcal{H}_*$, $\delta \in \mathbb{C}^1_{rd}([t_0,\infty)_{\mathbb{T}}, \mathbb{R}^+)$. Assume* (2.1) *and* (2.2) *hold, and $x(t)$ is a solution of* Eq. (1.1) *such that $x(t) > 0$ on a nonempty interval $(d, c] \cap \mathbb{T}$ with $d, c \in \mathbb{T}$. Let $w(t)$ be defined by* (2.4)*. Then*

$$\int_d^c H(\sigma(s),d)\delta^\sigma(s)\psi(s)\Delta s \le -H(c,d)w(c) + \eta_2(d) + \frac{1}{4}\int_{\sigma(d)}^c g(s)H(\sigma(s),d)\phi_2^2(c,s)\Delta s, \tag{3.2}$$

where $\eta_2(t)$ is defined by (2.22)*.*

Proof. As in the proof of Theorem 2.2 we obtain inequality (2.24). Then (3.2) follows directly from (2.24) with $t_\alpha = d$ and $t = c$. □

In the rest of this paper, we use the notation $\mathfrak{R} = \mathcal{H}_* \cap \mathcal{H}^*$.

Theorem 3.3. *Let $d, b, c \in \mathbb{T}$ such that $d < c < b$, and let* (2.1) *and* (2.2) *hold. Assume that for some $H \in \mathfrak{R}$ and $\delta \in \mathbb{C}^1_{rd}([t_0,\infty)_{\mathbb{T}}, \mathbb{R}^+)$,*

$$\begin{aligned}\frac{1}{H(c,d)}\int_d^c H(\sigma(s),d)\delta^\sigma(s)\psi(s)\Delta s &+ \frac{1}{H(b,c)}\int_c^b H(b,\sigma(s))\delta^\sigma(s)\psi(s)\Delta s \\ &> \frac{1}{4H(c,d)}\int_{\sigma(d)}^c g(s)H(\sigma(s),d)\phi_2^2(c,s)\Delta s \\ &+ \frac{1}{4H(b,c)}\int_c^{\rho(b)} g(s)H(b,\sigma(s))\phi_1^2(b,s)\Delta s \\ &+ \frac{1}{H(c,d)}\eta_2(d) + \frac{H_2^\Delta(b,\rho(b))}{H(b,c)}\eta(\rho(b))\chi_{\mu(\rho(b))},\end{aligned} \tag{3.3}$$

where $\psi(s)$ and $g(s)$ are the same as in Theorem 2.2 *and $\eta_2(t)$ is defined by* (2.22)*. Then every solution of* Eq. (1.1) *has at least one generalized zero in (d, b).*

Proof. Suppose the contrary. Then without loss of generality we may assume there exists a solution $x(t)$ of Eq. (1.1) such that $x(t) > 0$ for $t \in (d, b)$ with $d \geq T$. By Lemma 3.1 and 3.2 we see both the inequalities (3.1) and (3.2) hold, by dividing (3.1) and (3.2) by $H(b, c)$ and $H(c, d)$, respectively, and then adding, we have

$$\begin{aligned}
&\frac{1}{H(c,d)}\int_d^c H(\sigma(s),d)\delta^\sigma(s)\psi(s)\Delta s + \frac{1}{H(b,c)}\int_c^b H(b,\sigma(s))\delta^\sigma(s)\psi(s)\Delta s \\
&\leq \frac{1}{4H(c,d)}\int_{\sigma(d)}^c g(s)H(\sigma(s),d)\phi_2^2(c,s)\Delta s + \frac{1}{4H(b,c)}\int_c^{\rho(b)} g(s)H(b,\sigma(s))\phi_1^2(b,s)\Delta s \\
&\quad + \frac{1}{H(c,d)}\eta_2(d) + \frac{H_2^\Delta(b,\rho(b))}{H(b,c)}\eta(\rho(b))\chi_{\mu(\rho(b))},
\end{aligned}$$

which contradicts (3.3). □

Theorem 3.4. *Assume* (2.1) *and* (2.2) *hold.* Eq. (1.1) *is oscillatory provided that for any $T_* > T$, there exists $H \in \mathfrak{R}$ and $d, b, c \in \mathbb{R}$ such that $T_* \leq d < c < b$ and* (3.3) *holds.*

Proof. Pick a sequence $T_i \subset \mathbb{T}$ such that $T_i \to \infty$ as $i \to \infty$. By the assumption, for each $i \in \mathbb{N}$ there exists $d_i, b_i, c_i \in \mathbb{R}$ such that $T_i \leq d_i < c_i < b_i$, and (3.3) holds where d, b, c are replaced by d_i, b_i, c_i, respectively. From Theorem 3.1 every solution $x(t)$ has at least one generalized zero $t_i \in (d_i, b_i)$. Noting that $t_i > d_i \geq T_i$, $i \in \mathbb{N}$, we see that every solution has arbitrarily large generalized zeros. Thus Eq. (1.1) is oscillatory. □

Corollary 3.5. *Let* (2.1) *and* (2.2) *hold. Assume there exists $H \in \mathfrak{R}$ and $\delta \in \mathbb{C}^1_{rd}([t_0,\infty)_{\mathbb{T}}, \mathbb{R}^+)$ such that for any $l \geq T (\in \mathbb{T})$,*

(3.4)
$$\limsup_{t\to\infty}\Big[\int_l^t H(\sigma(s),l)\delta^\sigma(s)\psi(s)\Delta s - \frac{1}{4}\int_{\sigma(l)}^t g(s)H(\sigma(s),l)\phi_2^2(t,s)\Delta s - \eta_2(l)\Big] > 0,$$

and

$$\begin{aligned}
\limsup_{t\to\infty}\Big[\int_l^t H(t,\sigma(s))\delta^\sigma(s)\psi(s)\Delta s - \frac{1}{4}\int_l^{\rho(t)} g(s)H(t,\sigma(s))\phi_1^2(t,s)\Delta s \\
-H_2^\Delta(t,\rho(t))\eta(\rho(t))\chi_{\mu(\rho(t))}\Big] > 0.
\end{aligned} \tag{3.5}$$

Then Eq. (1.1) *is oscillatory.*

Proof. For any $T_1 \geq T$, Let $d = T_1$, in (3.4), we choose $l = d$. Then there exists $c > d$ such that

(3.6)
$$\limsup_{t\to\infty}\Big[\int_d^c H(\sigma(s),d)\delta^\sigma(s)\psi(s)\Delta s - \frac{1}{4}\int_{\sigma(d)}^c g(s)H(\sigma(s),d)\phi_2^2(c,s)\Delta s - \eta_2(d)\Big] > 0.$$

In (3.5), we choose $l = c$. Then there exists $b > c$ such that

$$\limsup_{t\to\infty}\Big[\int_c^b H(b,\sigma(s))\delta^\sigma(s)\psi(s)\Delta s - \frac{1}{4}\int_c^{\rho(b)} g(s)H(b,\sigma(s))\phi_1^2(b,s)\Delta s$$
$$-H_2^\Delta(b,\rho(b))\eta(\rho(b))\chi_{\mu(\rho(b))}\Big] > 0. \tag{3.7}$$

Combining (3.6) and (3.7) we obtain (3.3). The conclusion thus follows from Theorem 3.4. □

Corollary 3.6. *Let* $\mathbb{T}_1$, $\mathbb{T}_2$ *be defined by* (2.13), *and for any* $l, t \in \mathbb{T}$, *let* $\mathbb{T}_1(l,t) = [l,t) \cap \mathbb{T}_1$ *and* $\mathbb{T}_2(l,t) = [l,t) \cap \mathbb{T}_2$. *Assume* (2.1) *and* (2.2) *hold, and there exists a constant* $m > 1$ *such that for any* $l > T(\in \mathbb{T})$,

$$\limsup_{n\to\infty}\Big[\int_l^t (\sigma(s)-l)^m\delta^\sigma(s)\psi(s)\Delta s - \frac{m^2}{4}\int_{\mathbb{T}_1(\sigma(l),t)} (s-l)^{m-2}r(s)\delta(s)ds$$
$$-\frac{m^2}{4}\sum_{\mathbb{T}_2(\sigma(l),t)}\frac{(\sigma(s)-l)^{2m-2}}{s(s-l)^m\delta^\sigma(s)}\sigma(s)r(s)\delta^2(s)\mu(s) - \eta_2(l)\Big] > 0, \tag{3.8}$$

and

$$\limsup_{n\to\infty}\Big[\int_l^t (t-\sigma(s))^m\delta^\sigma(s)\psi(s)\Delta s - \frac{m^2}{4}\int_{\mathbb{T}_1(l,\rho(t))} (t-s)^{m-2}r(s)\delta(s)ds$$
$$-\frac{m^2}{4}\sum_{\mathbb{T}_2(l,\rho(t))}\frac{(t-s)^{2m-2}}{s(t-\sigma(s))^m\delta^\sigma(s)}\sigma(s)\delta^2(s)r(s)\mu(s)$$
$$-\frac{1}{2}(t-\rho(t))^m\delta(\rho(t))r(\rho(t))\frac{\sigma(\rho(t))\delta^\Delta(\rho(t))}{\rho(t)\delta^\sigma(\rho(t))}\Big] > 0. \tag{3.9}$$

Then Eq. (1.1) *is oscillatory.*

Proof. Let

$$H(t,s) = (t-s)^m \text{ and } a(t) = -\frac{\sigma(t)\delta^\Delta(t)}{2t\delta^\sigma(t)}.$$

Then $H \in \mathfrak{R}$, and $H_2^\Delta(t,s)$ satisfies (2.15), and $H_1^\Delta(t,s)$ is defined by

$$H_1^\Delta(s,t_\alpha) = \begin{cases} m(s-t_\alpha)^{m-1}, & s \in T_1, \\ ((\sigma(s)-t_\alpha)^m - (s-t_\alpha)^m)/\mu(s), & s \in T_2. \end{cases}$$

Noting from the mean value theorem that for $s \in [t_\alpha, \infty) \cap T_2$ there exists $\xi(s) \in [s, \sigma(s)]$ such that

$$0 \leq H_1^\Delta(s,t_\alpha) = m(\xi(s)-t_\alpha))^{m-1} \leq m(\sigma(s)-t_\alpha)^{m-1},$$

and

$$H_2^{\Delta}(t,\rho(t))\mu(\rho(t))\chi_{t-\rho(t)} = -(t-\rho(t))^m.$$

Therefore, conditions (3.8) and (3.9) are satisfied and hence the conclusion follows from Corollary 3.1. □

4. APPLICATIONS TO DIFFERENCE EQUATIONS

Here, we apply the results in Section 2 and 3 to obtain the Kamenev-type and interval oscillation criteria for the difference equations (1.3)–(1.4).

4.1. **Oscillation for Eq. (1.3).** The first one theorem is direct consequence of Corollary 2.3.

Theorem 4.1. *Let*

$$\sum_{n=t_3}^{\infty}\frac{1}{r_n} = \infty, \tag{4.1}$$

and

$$\sum_{n=t_3}^{\infty}\tau_n p_n = \infty. \tag{4.2}$$

Assume that there exist a constant $m>1$*, a positive sequel* δ_n *and* $n_0 > t_3$ *such that*

$$\limsup_{n\to\infty}\frac{1}{n^m}\Bigg[\sum_{k=n_0}^{n-1}(n-k-1)^m\delta_{k+1}\Big(\frac{Lp_k\tau_k}{k+1}+\frac{1}{2}\big[\frac{(k+1)\delta_k^{\Delta}r_k}{k\delta_{k+1}}\big]^{\Delta}+\frac{1}{4}\frac{(k+1)(\delta_k^{\Delta})^2r_k}{k\delta_{k+1}^2}\Big) \\ -\frac{m^2}{4}\sum_{k=n_0}^{n-2}\frac{(n-k)^{2m-2}(k+1)}{(n-k-1)^m\delta_{k+1}k}\delta_k^2r_k-\frac{1}{2}n\delta_{n-1}r_{n-1}\frac{\delta_{n-1}^{\Delta}}{(n-1)\delta_n}\Bigg]=\infty. \tag{4.3}$$

Then Eq. (1.3) *is oscillatory.*

The next theorem is derived from Corollary 3.6.

Theorem 4.2. *Let* (4.1) *and* (4.2) *hold. Assume there exist a constant* $m>1$ *and a positive sequel* δ_n *such that for every* $n_0 > t_3$*,*

$$\limsup_{n\to\infty}\Bigg[\sum_{k=n_0}^{n-1}(k+1-n_0)^m\delta_{k+1}\Big(\frac{Lp_k\tau_k}{k+1}+\frac{1}{2}\big(\frac{(k+1)\delta_k^{\Delta}r_k}{k\delta_{k+1}}\big)^{\Delta}+\frac{1}{4}\frac{(k+1)(\delta_k^{\Delta})^2r_k}{k\delta_{k+1}^2}\Big) \\ -\frac{m^2}{4}\sum_{k=n_0+1}^{n-1}\frac{(k+1-n_0)^{2m-2}(k+1)\delta_k^2}{(k-n_0)^mk\delta_{k+1}}r_k-\frac{m^2(n_0+1)r_{n_0}\delta_{n_0}^2}{4n_0\delta_{n_0+1}}\Bigg]>0, \tag{4.4}$$

and

$$\limsup_{n\to\infty}\Big[\sum_{k=n_0}^{n-1}(n-k-1)^m\delta_{k+1}\Big(\frac{Lp_k\tau_k}{k+1}+\frac{1}{2}\big(\frac{(k+1)\delta_k^{\Delta}r_k}{k\delta_{k+1}}\big)^{\Delta}+\frac{1}{4}\frac{(k+1)(\delta_k^{\Delta})^2r_k}{k\delta_{k+1}^2}\Big)$$

$$-\frac{m^2}{4}\sum_{k=n_0}^{n-2}\frac{(n-k)^{2m-2}(k+1)\delta_k^2}{(n-k-1)^mk\delta_{k+1}}r_k-\frac{n\delta_{n-1}r_{n-1}\delta_{n-1}^{\Delta}}{2(n-1)\delta_n}\Big]>0. \tag{4.5}$$

Then Eq. (1.3) *is oscillatory.*

Theorem 4.3. *Let $l,j\in\mathbb{T}$,* (4.1) *and* (4.2) *hold. Assume that for any $T\geq t_3$ there exists $T\leq l<j$ such that*

$$\sum_{n=l}^{j-1}(n+1)\Big[\frac{Lp\tau}{n+1}+\frac{1}{2}\big(\frac{r_n}{n}\big)^{\Delta}+\frac{r_n}{4n(n+1)}\Big]>0. \tag{4.6}$$

Then Eq. (1.3) *is oscillatory.*

Example 4.1. Consider the following equation

$$\Delta^2x+\frac{\beta}{\tau^2(t)}x(\tau(t))=0 \tag{4.7}$$

on the time scale $t\in[1,\infty)\cap\mathbb{T}$, where $\beta>1/4$ is a constant, $r(t)\equiv1$, $\tau(t)=\rho(t)$ and $p=\beta/\tau^2(t)$, $f(u)=u$. It is clear that (2.1) and (2.2) hold, and $L=1$. Let

$$a(t)=-\frac{1}{2t},\ \delta(t)=t,\ H(\sigma(t),t)=0,\ H(\sigma(t),s)=1,$$

for $t,s\in[1,\infty)\cap\mathbb{T},t\geq s$. Then, the left side of (2.3) takes the form

$$\limsup_{t\to\infty}\int_{t_1}^{t}\sigma(s)\Big[\frac{\beta\tau(s)}{\tau^2(s)\sigma(s)}+\frac{1}{2}\big(\frac{1}{s}\big)^{\Delta}+\frac{1}{4s\sigma(s)}\Big]\Delta s=\limsup_{t\to\infty}\int_{t_1}^{t}\Big[\frac{\beta}{\tau(s)}-\frac{1}{4s}\Big]\Delta s.$$

So, when $\mathbb{T}=[1,\infty)$ and $\beta>1/4$, by Theorem 2.2, Eq. (4.7) is oscillation.

On the other hand, when $\mathbb{T}=\mathbb{N}$ and $\beta>1/8$, Eq. (4.7) is oscillation by Theorem 4.1.

Example 4.2. Consider the equation

$$\Delta(r_n\Delta x_n)+p_nf(x_{\tau_n})=0, \tag{4.8}$$

on the time scale $\mathbb{T}=\bigcup_{n=1}^{\infty}[2n,2n+3/2]$, where

$$r_n=\begin{cases}e^{-3n}, & t\in[2n,2n+1],\\ >0, & t\in[2n+1,2n+3/2],\end{cases}\qquad p_n=\begin{cases}3e^{t-3n}, & t\in[2n,2n+1],\\ \geq\beta, & t\in[2n+1,2n+3/2],\end{cases}$$

$n\in\mathbb{N}$, $\tau(t)=\rho(t)$, $\beta>0$ is a constant. We next show that Eq. (4.8) is oscillatory.

Indeed, note that (2.1) and (2.2) hold. Let $d=2n$, $c=2n+1/2$, $b=2n+1$, and define

$$a(t)=-\frac{1}{2t},\ \delta(t)=t,\ H(t,s)=1,\ H(\sigma(t),t)=0,\ H(s,\sigma(s))=0,$$

for $t, s \in [2n, 2n+1]$, and $H \in \mathfrak{R}$.

Obviously, $\sigma(s) = s$, $\rho(s) = s$ for $s \in (2n, 2n+1)$. Then, by a direct computation, we find that, for any $n \in \mathbb{N}$,

$$\int_d^c H(\sigma(s), \sigma(l))\delta^\sigma(s)\psi(s)\Delta s = \int_{2n}^{2n+1/2} \Big(3se^{s-3n} - \frac{e^{-3n}}{4s}\Big) ds > 0, \tag{4.9}$$

and

$$\int_c^b H(\sigma(b), \sigma(d)\delta^\sigma(s)\psi(s)\Delta s = \int_{2n+1/2}^{2n+1} \Big(3se^{s-3n} - \frac{e^{-3n}}{4s}\Big) ds > 0, \tag{4.10}$$

and note that $H_1^\Delta(t, s) = 0$ and $H_2^\Delta(t, s) = 0$. Therefore, (4.9) and (4.10) imply that (3.3) holds, Hence the conclusion follows from Theorem 3.2.

4.2. **Oscillation for Eq. (1.4).** The following two theorems can be easily got from Corollaries 2.1 and 3.2 with the time scale $\mathbb{T} = q^{\mathbb{N}}$, $q > 1$.

Theorem 4.4. *Let*

$$\sum_{k=n_1}^{\infty} \frac{\mu(q^k)}{r(q^k)} = \infty, \tag{4.11}$$

and

$$\sum_{k=n_1}^{\infty} \mu(q^k)\tau(q^k)p(q^k) = \infty. \tag{4.12}$$

hold. Assume that there exist a constant $m > 1$, a positive sequel $\delta(n)$ and $n_0 > n_1$ such that

$$\limsup_{n\to\infty} \frac{1}{q^{nm}} \Big[\sum_{k=n_0}^{n-1} (q^n - q^{k+1})^m \delta(q^{k+1}) q^k (q-1) \Big(\frac{Lp(q^k)\tau(q^k)}{q^{k+1}} + \frac{1}{2}\big[\frac{q\delta^\Delta(q^k)r(q^k)}{\delta(q^{k+1})}\big]^\Delta$$
$$+ \frac{1}{4} qr(q^k)\big[\frac{\delta^\Delta(q^k)}{\delta(q^{k+1})}\big]^2\Big) - \frac{m^2}{4} \sum_{k=n_0}^{n-2} \frac{(q^n - q^k)^{2m-2} q^{k+1}(q-1)\delta^2(q^k)}{(q^n - q^{k+1})^m \delta(q^{k+1})} r(q^k)$$
$$- \frac{1}{2}\delta(q^{n-1})r(q^{n-1})q^{m(n-1)+1}(q-1)^m \frac{\delta^\Delta(q^{n-1})}{\delta(q^n)}\Big] = \infty.$$

Then Eq. (1.4) *is oscillatory.*

Theorem 4.5. *Let* (4.11) *and* (4.12) *hold. Assume that there exist a constant $m > 1$ such that for any $n_0 > n_1$,*

$$\limsup_{n\to\infty} \Big[\sum_{k=n_0}^{n-1} (q^{k+1} - q^{n_0})^m \delta(q^{k+1}) q^k (q-1) \Big(\frac{Lp(q^k)\tau(q^k)}{q^{k+1}} + \frac{1}{2}\big[\frac{q\delta^\Delta(q^k)r(q^k)}{\delta(q^{k+1})}\big]^\Delta$$
$$+ \frac{1}{4} qr(q^k)\big[\frac{\delta^\Delta(q^k)}{\delta(q^{k+1})}\big]^2\Big) - \frac{m^2}{4} \sum_{k=n_0+1}^{n-1} \frac{(q^{k+1} - q^{n_0})^{m-2}\delta^2(q^k)q^{k+1}(q-1)}{\delta(q^{k+1})(q^k - q^{n_0})^m} r(q^k)$$
$$- \frac{m^2 q^{n_0(m-1)+1}(q-1)^{m-1} r(q^{n_0})\delta^2(q^{n_0})}{4\delta(q^{n_0+1})}\Big] > 0,$$

and

$$\limsup_{n\to\infty}\Big[\sum_{k=n_0}^{n-2}(q^n-q^{k+1})^m\delta(q^{k+1})q^k(q-1)\Big(\frac{Lp(q^k)\tau(q^k)}{q^{k+1}}+\frac{1}{2}\big[\frac{q\delta^{\Delta}(q^k)r(q^k)}{\delta(q^{k+1})}\big]^{\Delta}$$
$$+\frac{1}{4}qr(q^k)\big[\frac{\delta^{\Delta}(q^k)}{\delta(q^{k+1})}\big]^2\Big)-\frac{m^2}{4}\sum_{k=n_0}^{n-1}\frac{(q^n-q^k)^{2m-2}q^{k+1}(q-1)\delta^2(q^k)}{(q^n-q^{k+1})^m\delta(q^{k+1})}r(q^k)$$
$$-\frac{1}{2}q^{m(n-1)+1}(q-1)^m\delta(q^{n-1})r(q^{n-1})\frac{\delta^{\Delta}(q^{n-1})}{\delta(q^n)}\Big]>0.$$

Then Eq. (1.4) *is oscillatory.*

Theorem 4.6. *Let $l,j\in\mathbb{N}$,* (4.11) *and* (4.12) *hold. Assume that, for any $T\geq q^{n_1}$, there exists $T\leq l<j$ such that*

$$\sum_{n=l}^{j-1}\Big[Lp\tau+\frac{1}{2}q^{n+1}\big(\frac{r(q^n)}{q^n}\big)^{\Delta}+\frac{r(q^n)}{4q^n}\Big]>0. \tag{4.13}$$

Then Eq. (1.4) *is oscillatory.*

Example 4.3. Consider the equation

$$\Delta_q\big(\frac{1}{t}\Delta_q x(t)\big)+tx(\tau(t))=0 \tag{4.14}$$

where $\mathbb{T}=q^{\mathbb{N}}$, $n\in\mathbb{N}$, $q>1$, $r(t)=1/t$ and $p(t)=t$, $\tau(t)=\rho(t)$ for $t\in\mathbb{T}$. We then show that Eq. (4.14) is oscillatory.

In fact, let $m=2$, $\delta(t)=1$, $a(t)=0$. Here, $L=1$, then

$$\limsup_{n\to\infty}\frac{1}{(q^n)^2}\Big[\sum_{k=n_0}^{n-1}(q^n-q^{k+1})^2\frac{q^{2k-1}}{q^{k+1}}(q^{k+1}-q^k)-\sum_{k=n_0}^{n-2}\frac{(q^n-q^k)^2q(q^{k+1}-q^k)}{(q^n-q^{k+1})^2q^k}\Big]$$
$$=q(q-1)\limsup_{n\to\infty}\Big[\sum_{k=n_0}^{n-1}(1-q^{k+1-n})^2q^{2k-3}-\sum_{k=n_0}^{n-2}\frac{(1-q^{k-n})^2}{(q^n(1-q^{k+1-n}))^2}\Big]=\infty.$$

Hence the conclusion follows from Theorem 4.4.

Example 4.4. Consider the equation

$$\Delta_q(\Delta_q x(t))+\frac{e^t}{\tau(t)}x(\tau(t))=0 \tag{4.15}$$

on the time scales $t\in q^{\mathbb{N}}$, where $\tau(t)=\rho(t)$. We then claim that Eq (4.15) is oscillatory.

Indeed, it is a clear that (4.11), (4.12) hold. For any $n_0\geq n_1$, there exists $q^{n_0}\leq d<c<b$, by a simple computation, we get

$$\sum_{s=d}^{s=b}\frac{q-1}{s}\big(se^s-\frac{1}{4}\big)>0.$$

i.e., (4.13) holds. Hence the conclusion follows from Theorem 4.6.

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