ISSN: 1748-0345 (Online) www.tagajournal.com

THREE SOLUTIONS FOR A CLASS OF GRADIENT KIRCHHOFF-TYPE SYSTEMS DEPENDING ON TWO PARAMETERS

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ABSTRACT. In this paper we shall discuss the existence of at least three solutions for the class of two-point boundary value Kirchhoff-type systems

$$\begin{cases} -K_i(\int_a^b |u_i'(x)|^2 dx)u_i'' = \lambda F_{u_i}(x, u_1, \dots, u_n) + \mu G_{u_i}(x, u_1, \dots, u_n), \\ u_i(a) = u_i(b) = 0 \end{cases}$$

for $1 \leq i \leq n$. The approach is fully based on a recent three critical points theorem of B. Ricceri [A three critical points theorem revisited, Nonlinear Anal. 70/9 (2009) 3084–3089].

AMS (MOS) Subject Classification. 34B15, 47J10.

1. INTRODUCTION

In the literature many results focus on the existence of multiple solutions to boundary value problems. There seems to be increasing interest in multiple solutions to boundary value problems, because of their applications in physical processes described by differential equations can exhibit more than one solution, and other fields. For example, certain chemical reactions in tubular reactors can be mathematically described by a nonlinear two-point boundary value problem and one is interested if multiple steady-states to the problem exist. For a instance treatment of chemical reactor theory and multiple solutions see [2, section 7] and the references therein. For additional approaches to the existence of multiple solutions to boundary value problems, see [11, 12] and references therein. Moreover, in [19], Ricceri obtained a three critical points theorem and in [18] gave a general version of the theorem to extend the results for a class of more extensive equations. By these results, many authors studied the existence of at least three solutions for BVPs (for instance, see [1, 5, 6, 9, 10, 13, 21]).

In this paper, we are interested in establishing the existence of three (weak) solutions for the following Kirchhoff-type system on a bounded interval $[a,b]$ in $\mathbb{R}$ $(a<b)$

$$(1.1)\qquad \begin{cases} -K_i(\int_a^b |u_i'(x)|^2dx)u_i'' = \lambda F_{u_i}(x,u_1,\ldots,u_n) + \mu G_{u_i}(x,u_1,\ldots,u_n), \\ u_i(a)=u_i(b)=0, \quad 1\le i\le n \end{cases}$$

where $K_i : [0,+\infty[\to \mathbb{R}$ for $1\le i\le n$ are n continuous function, $\lambda,\ \mu$ are two positive parameters, $F:[a,b]\times\mathbb{R}^n\to\mathbb{R}$ is a measurable function with respect to x in $[a,b]$ for every $(t_1,\ldots,t_n)\in\mathbb{R}^n$, and is a C^1-function with respect to $(t_1,\ldots,t_n)\in\mathbb{R}^n$ for every x in $[a,b]$ such that $F(x,0,\ldots,0)=0$ for every $x\in[a,b]$, and $G:[a,b]\times\mathbb{R}^n\to\mathbb{R}$ is a measurable function with respect to x in $[a,b]$ for every $(t_1,\ldots,t_n)\in\mathbb{R}^n$, and is a C^1-function with respect to $(t_1,\ldots,t_n)\in\mathbb{R}^n$ for every x in $[a,b]$ and satisfied the condition

$$(1.2)\qquad \sup_{|(t_1,\ldots,t_n)|\le s} \sum_{i=1}^n |G_{t_i}(x,t_1,\ldots,t_n)| \le m_s(x)$$

for all $s>0$ and some $m_s\in L^1$ with $G(.,0,\ldots,0)\in L^1$, and F_t and G_t denote the partial derivative of F and G with respect to t, respectively using Ricceri's three critical points theorem.

Problems of Kirchhoff-type have been widely investigated. We refer the reader to the papers [3, 7, 8, 14, 15, 16, 23] and the references therein. B. Ricceri in an interesting paper [20] established the existence of at least three weak solutions to a class of Kirchhoff-type doubly eigenvalue boundary value problem using Theorem 2 of [17].

We mean by a (weak) solution of system (1.1), any $u=(u_1,\ldots,u_n)\in(W_0^{1,2}([a,b]))^n$ such that

$$\sum_{i=1}^n K_i(\int_a^b |u_i'(x)|^2dx)\int_a^b u_i'(x)v_i'(x)dx - \lambda\int_a^b\sum_{i=1}^n F_{u_i}(x,u_1(x),\ldots,u_n(x))v_i(x)dx$$

$$-\mu\int_a^b\sum_{i=1}^n G_{u_i}(x,u_1(x),\ldots,u_n(x))v_i(x)dx = 0$$

for every $v=(v_1,\ldots,v_n)\in(W_0^{1,2}([a,b]))^n$.

Our main result is Theorem 2.1, in the next section. Its proof is fully based on a very recent three critical points theorem of B. Ricceri [18] (see also [4] for related results) that we recall here for the reader's convenience.

Theorem 1.1 ([18]). *Let X be a reflexive real Banach space, $I\subseteq\mathbb{R}$ an interval, $\Phi : X \longrightarrow \mathbb{R}$ a sequentially weakly lower semicontinuous C^1 functional, bounded on* *each bounded subset of X, whose derivative admits a continuous inverse on X^* and $J : X \longrightarrow \mathbb{R}$ a C^1 functional with compact derivative. Assume that*

$$\lim_{||x|| \to +\infty} (\Phi(x) + \lambda J(x)) = +\infty$$

for all $\lambda \in I$, and that there exists $\rho \in \mathbb{R}$ such that

$$\sup_{\lambda \in I} \inf_{x \in X} (\Phi(x) + \lambda(J(x) + \rho)) < \inf_{x \in X} \sup_{\lambda \in I} (\Phi(x) + \lambda(J(x) + \rho)).$$

Then, there exist a non-empty open set interval $A \subseteq I$ and a positive real number γ with the following property: for every $\lambda \in A$ and every C^1 functional $\Psi : X \longrightarrow \mathbb{R}$ with compact derivative, there exists $\delta > 0$ such that, for each $\mu \in [0, \delta]$, the equation

$$\Phi'(u) + \lambda J'(u) + \mu \Psi'(u) = 0$$

has at least three solutions in X whose norms are less than γ.

For using later, we also recall the following result, Proposition 1.3 of [4] with J replaced by $-J$, precisely to show the minimax inequality in Theorem 1.1.

Proposition 1.2 ([4]). *Let X be a non-empty set and Φ, J two real functions on X. Assume that $\Phi(u) \geq 0$ for every $u \in X$ and there exists $u_0 \in X$ such that $\Phi(u_0) = J(u_0) = 0$. Further, assume that there exist $u_1 \in X$, $r > 0$ such that $\Phi(u_1) > r$ and*

$$\sup_{\Phi(u) < r} (-J(u)) < r \frac{-J(u_1)}{\Phi(u_1)}.$$

Then, for every $h > 1$ and for every $\rho \in \mathbb{R}$ satisfying

$$\sup_{\Phi(u) < r} (-J(u)) + \frac{r \frac{-J(u_1)}{\Phi(u_1)} - \sup_{\Phi(u) < r}(-J(u))}{h} < \rho < r \frac{-J(u_1)}{\Phi(u_1)},$$

one has

$$\sup_{\lambda \in R} \inf_{u \in X} (\Phi(u) + \lambda(J(u) + \rho)) < \inf_{u \in X} \sup_{\lambda \in [0, v]} (\Phi(u) + \lambda(J(u) + \rho))$$

where

$$v = \frac{hr}{r \frac{-J(u_1)}{\Phi(u_1)} - \sup_{\Phi(u) < r}(-J(u))}.$$

For other basic notations and definitions, we refer the reader to [22].

The rest of the paper is organized as follows: Section 2 contains the statements of our results, proofs of the corollaries and an example to illustrate the results. Section 3 consists the proof of our main result.

2. MAIN RESULT AND SOME CONSEQUENCES

Let $K_i : [0, +\infty[\to \mathbb{R}$ for $1 \leq i \leq n$ be n continuous function such that there exist n positive number m_i with $K_i(t) \geq m_i$ for all $t \geq 0$ and let $F : [a, b] \times \mathbb{R}^n \to \mathbb{R}$ be a measurable function with respect to x in $[a, b]$ for every $(t_1, \ldots, t_n) \in \mathbb{R}^n$, and a C^1-function with respect to $(t_1, \ldots, t_n) \in \mathbb{R}^n$ for almost every x in $[a, b]$ such that $F(x, 0, \ldots, 0) = 0$ for every $x \in [a, b]$. Corresponding to K_i we introduce the functions $\tilde{K}_i : [0, +\infty[\to \mathbb{R}$ as follows

$$\tilde{K}_i(t) = \int_0^t K_i(s)ds \text{ for all } t \geq 0$$

for $1 \leq i \leq n$.

We formulate our main result as follows:

Theorem 2.1. *Assume that there exist a positive constant r and a function $w = (w_1, \ldots, w_n) \in (W^{1,2}([a, b]))^n$ such that*

(i) $\sum_{i=1}^n \tilde{K}_i(\int_a^b |w_i'(x)|^2 dx) > 2r$;

(ii) $M_1 := 2r \frac{\int_a^b F(x,w(x))dx}{\sum_{i=1}^n \tilde{K}_i(\int_a^b |w_i'(x)|^2 dx)} - \int_a^b \sup_{(t_1,\ldots,t_n)\in A_1} F(x, t_1, \ldots, t_n)dx > 0$ *where* $A_1 := \left\{(t_1, \ldots, t_n) \mid \sum_{i=1}^n t_i^2 \leq \frac{r(b-a)}{2\min\{m_i, 1\leq i\leq n\}}\right\}$;

(iii) $\limsup_{(|t_1|,\ldots,|t_n|)\to(+\infty,\ldots,+\infty)} \frac{F(x,t_1,\ldots,t_n)}{\sum_{i=1}^n t_i^2} < \frac{2\min\{m_i,1\leq i\leq n\}}{\tau(b-a)^2}$ *uniformly with respect to* $x \in [a, b]$ *for some* τ *satisfying*

$$\tau > \frac{r}{M_1}.$$

Further, assume that there exist n continuous functions $h_i : [0, +\infty[\to \mathbb{R}$ for $1 \leq i \leq n$ such that

$$h_i(tK_i(t^2)) = t$$

for all $t \geq 0$.

Then, there exist a non-empty open set $A \subseteq]0, \tau[$ and a real number $\gamma > 0$ with the following property: for every $\lambda \in A$ and for an arbitrary function $G : [a, b] \times \mathbb{R}^n \to \mathbb{R}$ which is measurable in $[a, b]$ and C^1 in $\mathbb{R}^n$ satisfying (1.2), there exists $\delta > 0$ such that, for each $\mu \in [0, \delta]$, the system (1.1) has at least three weak solutions in $(W_0^{1,2}([a, b]))^n$ whose norms are less than γ.

Now we want to present a verifiable consequence of the main result where the test function w is specified.

Corollary 2.2. *Assume that there exist $2n+2$ positive constants c_i, d_i for $1 \leq i \leq n$, α and β with $\beta + \alpha < b - a$ such that*

(i′) $\sum_{i=1}^n \tilde{K}_i(\frac{\alpha+\beta}{\alpha\beta} \sum_{i=1}^n d_i^2) > \frac{4}{b-a} \sum_{i=1}^n m_i c_i^2$;

(ii′) $M_2 := \frac{4}{b-a}\sum_{i=1}^n m_i c_i^2 \frac{B}{\sum_{i=1}^n \tilde{K}_i(\int_a^b |w_i'(x)|^2 dx)} - \int_a^b \sup_{(t_1,\dots,t_n)\in A_2} F(x,t_1,\dots,t_n)dx > 0$ *where*

$$B := \int_a^{a+\alpha} F\left(x, \frac{d_1}{\alpha}(x-a),\dots,\frac{d_n}{\alpha}(x-a)\right) + \int_{a+\alpha}^{b-\beta} F(x,d_1,\dots,d_n) + \int_{b-\beta}^{b} F\left(x, \frac{d_1}{\beta}(b-x),\dots,\frac{d_n}{\beta}(b-x)\right)$$

and $A_2 := \{(t_1,\dots,t_n) \mid \sum_{i=1}^n t_i^2 \le \frac{\sum_{i=1}^n m_i c_i^2}{\min\{m_i,\ 1\le i\le n\}}\}$;

(iii′) $\limsup_{(|t_1|,\dots,|t_n|)\to(+\infty,\dots,+\infty)} \frac{F(x,t_1,\dots,t_n)}{\sum_{i=1}^n t_i^2} < \frac{2\min\{m_i, 1\le i\le n\}}{\tau(b-a)^2}$ *uniformly with respect to* $x\in[a,b]$ *for some* τ *satisfying*

$$\tau > \frac{4\sum_{i=1}^n m_i c_i^2}{(b-a)M_2}.$$

Further, assume that there exist n *continuous functions* $h_i : [0,+\infty[\to \mathbb{R}$ *for* $1\le i\le n$ *such that*

$$h_i(tK_i(t^2)) = t$$

for all $t \ge 0$.

Then, there exist a non-empty open set $A \subseteq]0,\tau[$ *and a real number* $\gamma > 0$ *with the following property: for every* $\lambda \in A$ *and for an arbitrary function* $G : [a,b]\times\mathbb{R}^n \to \mathbb{R}$ *which is measurable in* $[a,b]$ *and* C^1 *in* $\mathbb{R}^n$ *satisfying* (1.2)*, there exists* $\delta > 0$ *such that, for each* $\mu \in [0,\delta]$*, the system* (1.1) *has at least three weak solutions in* $(W_0^{1,2}([a,b]))^n$ *whose norms are less than* γ.

Proof. Set $w(x) = (w_1(x),\dots,w_n(x))$ such that for $1\le i\le n$,

$$w_i(x) = \begin{cases} \frac{d_i}{\alpha}(x-a) & \text{if } a \le x < a+\alpha, \\ d_i & \text{if } a+\alpha \le x \le b-\beta, \\ \frac{d_i}{\beta}(b-x) & \text{if } b-\beta < x \le b \end{cases} \tag{2.1}$$

and $r = \frac{2}{b-a}\sum_{i=1}^n m_i c_i^2$ where constants c_i, d_i, α and β are given in the statement of the theorem. It is clear from (2.1) that $w \in X$ and, in particular, one has

$$\|w_i\|^2 = d_i^2\left(\frac{\alpha+\beta}{\alpha\beta}\right) \tag{2.2}$$

for $1\le i\le n$. Moreover with this choice of w and taking into account (2.2), from (i′) and (ii′) we get (i) and (ii), respectively. Hence, using Theorem 2.1, we have the desired conclusion. □

We want to point out a remarkable particular situation of Corollary 2.2.

Corollary 2.3. *Fix* $p_i,\ q_i > 0$ *for* $1\le i\le n$*. Assume that there exist* $2n+2$ *positive constants* c_i, d_i *for* $1\le i\le n$, α *and* β *with* $\beta+\alpha < b-a$ *such that*

(i″) $\sum_{i=1}^n[p_i d_i^2(\frac{\alpha+\beta}{\alpha\beta}) + \frac{q_i}{2}d_i^4(\frac{\alpha+\beta}{\alpha\beta})^2] > \frac{4}{b-a}\sum_{i=1}^n p_i c_i^2$;

(ii″) $M_3 := \frac{4}{b-a}\sum_{i=1}^n p_i c_i^2 \frac{B}{\sum_{i=1}^n [p_i d_i^2(\frac{\alpha+\beta}{\alpha\beta}) + \frac{q_i}{2} d_i^4(\frac{\alpha+\beta}{\alpha\beta})^2]} - \int_a^b \sup_{(t_1,\ldots,t_n)\in A_3} F(x,t_1,\ldots,t_n)dx > 0$ *where* B *is given as in* (*ii*′) *and* $A_3 := \{(t_1,\ldots,t_n) | \sum_{i=1}^n t_i^2 \leq \frac{\sum_{i=1}^n p_i c_i^2}{\min\{p_i, 1\leq i\leq n\}}\}$;

(iii″) $\limsup_{(|t_1|,\ldots,|t_n|)\to(+\infty,\ldots,+\infty)} \frac{F(x,t_1,\ldots,t_n)}{\sum_{i=1}^n t_i^2} < \frac{2\min\{p_i, 1\leq i\leq n\}}{\tau(b-a)^2}$ *uniformly with respect to* $x\in[a,b]$ *for some* τ *satisfying*

$$\tau > \frac{4\sum_{i=1}^n p_i c_i^2}{(b-a)M_3}.$$

Then, there exist a non-empty open set $A \subseteq]0,\tau[$ *and a real number* $\gamma > 0$ *with the following property: for every* $\lambda \in A$ *and for an arbitrary function* $G : [a,b]\times\mathbb{R}^n \to \mathbb{R}$ *which is measurable in* $[a,b]$ *and* C^1 *in* $\mathbb{R}^n$ *satisfying* (1.2), *there exists* $\delta > 0$ *such that, for each* $\mu \in [0,\delta]$, *the system*

$$\text{(2.3)}\qquad \begin{cases} -(p_i + q_i\int_a^b |u_i'(x)|^2dx)u_i'' = \lambda F_{u_i}(x,u_1,\ldots,u_n) + \mu G_{u_i}(x,u_1,\ldots,u_n), \\ u_i(a) = u_i(b) = 0 \end{cases}$$

for $1 \leq i \leq n$, *has at least three weak solutions in* $(W_0^{1,2}([a,b]))^n$ *whose norms are less than* γ.

Proof. For fixed $p_i, q_i > 0$ for $1\leq i\leq n$, set $K_i(t) = p_i + q_i t$ for all $t \geq 0$. Bearing in mind that $m_i = p_i$ for $1\leq i\leq n$, from (i″), (ii″) and (iii″) we see that (i′), (ii′) and (iii′) hold respectively. Also we note that there exist n continuous function $h_i : [0,+\infty[\to \mathbb{R}$ for $1\leq i\leq n$ such that

$$h_i(tK_i(t^2)) = t$$

for all $t \geq 0$ because the functions K_i for $1\leq i\leq n$ are nondecreasing in $[0,+\infty[$ with $K_i(0) > 0$ and $t \to tK_i(t^2)$ $(t\geq 0)$ for $1\leq i\leq n$ are increasing and onto $[0,+\infty[$. Hence, Corollary 2.2 yields the conclusion. □

We present an example to illustrate our results applying by Corollary 2.3.

Example 2.4. Consider the system

$$\text{(2.4)}\qquad \begin{cases} -(\frac{1}{128} + \frac{1}{64}\int_0^1 |u_i'(x)|^2dx)u_i'' = \lambda(e^{-u_i^+}(u_i^+)^{\gamma_i-1}(\gamma_i - u_i^+) + 1) + \mu G_{u_i}(u_1,u_2), \\ u_i(0) = u_i(1) = 0, \quad 1\leq i\leq 2 \end{cases}$$

where $u_i^+ = \max\{u_i, 0\}$, $\gamma_i \geq 2$ for $i = 1,2$ are real numbers such that the following inequality holds

$$\text{(2.5)}\qquad \underline{M}_3 = \frac{1/16}{100/512 + 10^4}(10^{\min\{\gamma_1,\gamma_2\}}e^{-10} + 10) - 2^{\frac{\max\{\gamma_1,\gamma_2\}}{2}+1} - 2^{\frac{3}{2}} > 0,$$

λ, μ are two positive parameter and $G : \mathbb{R}^2 \to \mathbb{R}$ stands for any C^1-function.
Set $p_i = \frac{1}{128}$, $q_i = \frac{1}{64}$ for $i = 1,2$ and

$$F(t_1,t_2) = \begin{cases} 0 & \text{for all } t_i < 0, \\ \sum_{i=1}^2 (t_i^{\gamma_i}e^{-t_i} + t_i) & \text{for all } t_i \geq 0, \quad 1\leq i\leq 2 \end{cases}$$

for each $(t_1, t_2) \in \mathbb{R}^2$. Assumptions (i$''$) is satisfied by choosing, for instance $d_i = 10, c_i = 1$ for $i = 1, 2$, $[a, b] = [0, 1]$ and $\alpha = \beta = \frac{1}{4}$. For (ii$''$), taking (2.5) into account we see that

$$\begin{aligned} M_3 &= \frac{1/16}{100/512 + 10^4} B - \sup_{t_1^2 + t_2^2 \le 2} F(t_1, t_2) \\ &\ge \frac{1/16}{100/512 + 10^4} \frac{1}{2} \left(\sum_{i=1}^{2} d_i^{\gamma_i} e^{-d_i} + d_i \right) - [2^{\frac{\max\{\gamma_1, \gamma_2\}}{2} + 1} + 2^{\frac{3}{2}}] \\ &\ge \frac{1/16}{100/512 + 10^4} \left(d_i^{\min\{\gamma_1, \gamma_2\}} e^{-d_i} + d_i \right) - [2^{\frac{\max\{\gamma_1, \gamma_2\}}{2} + 1} + 2^{\frac{3}{2}}] \\ &= \frac{1/16}{100/512 + 10^4} \left(10^{\min\{\gamma_1, \gamma_2\}} e^{-10} + 10 \right) - [2^{\frac{\max\{\gamma_1, \gamma_2\}}{2} + 1} + 2^{\frac{3}{2}}] \\ &= \underline{M}_3 > 0. \end{aligned}$$

Moreover, since

$$\limsup_{(|t_1|, \ldots, |t_n|) \to (+\infty, \ldots, +\infty)} \frac{F(t_1, t_2)}{\sum_{i=1}^{2} t_i^2} = 0,$$

we clearly observe that Assumption (iii$''$) is fulfilled. Now we can apply Corollary 2.3 to the system (2.4) for any $\tau > \frac{1}{16\underline{M}_3}$.

Finally, we end this section by giving the following result.

Corollary 2.5. *Assume that there exist $n + 2$ positive constants c_i, d_i for $1 \le i \le n$, α and β with $\beta + \alpha < b - a$ such that*

(i$'''$) $\sum_{i=1}^{n} d_i^2 (\frac{\alpha+\beta}{\alpha\beta}) > \frac{4}{b-a} \sum_{i=1}^{n} c_i^2$;

(ii$'''$) $M_4 := \frac{4}{b-a} \sum_{i=1}^{n} c_i^2 \frac{B}{\sum_{i=1}^{n} d_i^2 (\frac{\alpha+\beta}{\alpha\beta})} - \int_a^b \sup_{(t_1, \ldots, t_n) \in A_4} F(x, t_1, \ldots, t_n) dx > 0$ *where B is given as in (ii$'$) and $A_4 := \{(t_1, \ldots, t_n) | \sum_{i=1}^{n} t_i^2 \le \sum_{i=1}^{n} c_i^2\}$;*

(iii$'''$) $\limsup_{(|t_1|, \ldots, |t_n|) \to (+\infty, \ldots, +\infty)} \frac{F(x, t_1, \ldots, t_n)}{\sum_{i=1}^{n} t_i^2} < \frac{2}{\tau(b-a)^2}$ *uniformly with respect to $x \in [a, b]$ for some τ satisfying*

$$\tau > \frac{4 \sum_{i=1}^{n} c_i^2}{(b-a) M_4}.$$

Then, there exist a non-empty open set $A \subseteq]0, \tau[$ and a real number $\gamma > 0$ with the following property: for every $\lambda \in A$ and for an arbitrary function $G : [a, b] \times \mathbb{R}^n \to \mathbb{R}$ which is measurable in $[a, b]$ and C^1 in $\mathbb{R}^n$ satisfying (1.2), there exists $\delta > 0$ such that, for each $\mu \in [0, \delta]$, the system

$$\begin{cases} -u_i'' = \lambda F_{u_i}(x, u_1, \ldots, u_n) + \mu G_{u_i}(x, u_1, \ldots, u_n), \\ u_i(a) = u_i(b) = 0 \end{cases}$$

for $1 \le i \le n$, has at least three weak solutions in $(W_0^{1,2}([a, b]))^n$ whose norms are less than γ.

Proof. Taking $p_i = 1$ and $q_i = 0$ for $1 \le i \le n$, all the assumptions of Corollary 2.3 are fulfilled. Hence, from Corollary 2.3 we have the conclusion. □

3. PROOF OF THE MAIN RESULT

In order to apply Theorem 1.1 to our problem, let X be the Cartesian product of the n Sobolev spaces $W_0^{1,2}([a,b]), \dots, W_0^{1,2}([a,b])$, i.e., $X = (W_0^{1,2}([a,b]))^n$ equipped with the norm

$$\|(u_1, u_2, \dots, u_n)\| = \sum_{i=1}^{n} \|u_i\|$$

where $\|u_i\| = \left(\int_a^b (|u_i'(x)|^2)dx\right)^{1/2}$ for $1 \le i \le n$.

We introduce the functionals $\Phi,\ J : X \to \mathbb{R}$ for each $u = (u_1, \dots, u_n) \in X$, as follows

$$\Phi(u) = \frac{1}{2}\sum_{i=1}^{n} \tilde{K}_i(\|u_i\|^2) \tag{3.1}$$

and

$$J(u) = -\int_a^b F(x, u_1(x), \dots, u_n(x))dx. \tag{3.2}$$

Let us prove that the functionals Φ and J satisfy the required conditions. It is well known that J is a differentiable functional whose differential at the point $u \in X$ is

$$J'(u)(v) = -\int_a^b \sum_{i=1}^{n} F_{u_i}(x, u_1(x), \dots, u_n(x))v_i(x)dx$$

for every $v = (v_1, \dots, v_n) \in X$. We claim that J' is a compact operator. Indeed, for fixed $(u_1, \dots, u_n) \in X$, assume $(u_{1m}, \dots, u_{nm}) \to (u_1, \dots, u_n)$ weakly in X as $m \to +\infty$. Then $(u_{1m}, \dots, u_{nm}) \to (u_1, \dots, u_n)$ strongly in $C([a,b])$. Since $F(x, ., \dots, .)$ is C^1 in $\mathbb{R}^n$ for every $x \in [a,b]$, so it is continuous in $\mathbb{R}^n$ for every $x \in [a,b]$, and we get that $F(x, u_{1m}, \dots, u_{nm}) \to F(x, u_1, \dots, u_n)$ strongly as $m \to +\infty$. By the Lebesgue control convergence theorem, $J'(u_{1m}, \dots, u_{nm}) \to J'(u_1, \dots, u_n)$ strongly, which means that J' is strongly continuous, then it is a compact operator. Hence the claim is true. Moreover, it is well known that Φ is sequentially weakly lower semicontinuous as well as bounded on each bounded subset of X and continuously differentiable whose differential at the point $u \in X$ is

$$\Phi'(u)(v) = \sum_{i=1}^{n} K_i(\int_a^b |u_i'(x)|^2 dx) \int_a^b u_i'(x)v_i'(x)dx$$

for every $v \in X$. We claim that Φ' admits a continuous inverse on X (we identity X with X^*). To prove this fact, we need to find a continuous operator $T : X \to X$ such that $T(\Phi'(u)) = u$ for all $u \in X$. Let $T : X \to X$ be the operator defined by

$$T(v_1, \dots, v_n) = (T_1(v_1), \dots, T_n(v_n))$$

such that $T_i : W_0^{1,2}([a,b]) \to \mathbb{R}$ expressed by

$$T_i(v_i) = \begin{cases} \frac{h_i(\|v_i\|)}{\|v_i\|} v_i & \text{if } v_i \neq 0 \\ 0 & \text{if } v_i = 0, \end{cases}$$

where h_i is defined in the statement of the theorem for $1 \leq i \leq n$. Since, h_i is continuous and $h_i(0) = 0$, we see that the operator T_i is continuous in X for $1 \leq i \leq n$ and consequensly T in X. For every $u \in X$, taking into account that $\inf_{t \geq 0} K_i(t) \geq m_i > 0$, we have since $h_i(t\, K_i(t^2)) = t$ for all $t \geq 0$ that

$$\begin{aligned} T(\Phi'(u)) &= T\left(K_1(\|u_1\|^2)u_1, \ldots, K_n(\|u_n\|^2)u_n\right) \\ &= \left(T_1(K_1(\|u_1\|^2)u_1), \ldots, T_n(K_n(\|u_n\|^2)u_n)\right) \\ &= \left(\frac{h_1(K_1(\|u_1\|^2)\|u_1\|)}{K_1(\|u_1\|^2)\|u_1\|} K_1(\|u_1\|^2)u_1, \ldots, \frac{h_n(K_n(\|u_n\|^2)\|u_n\|)}{K_n(\|u_n\|^2)\|u_n\|} K_n(\|u_n\|^2)u_n\right) \\ &= \left(\frac{\|u_1\|}{K_1(\|u_1\|^2)\|u_1\|} K_1(\|u_1\|^2)u_1, \ldots, \frac{\|u_n\|}{K_n(\|u_n\|^2)\|u_n\|} (\|u_n\|^2)u_n\right) \\ &= (u_1, \ldots, u_n). \end{aligned}$$

and so our claim is true. Moreover, since for $1 \leq i \leq n$, $m_i \leq K_i(s)$ for all $s \in [0, +\infty[$, from (3.1) we have

$$\Phi(u) \geq \frac{1}{2} \sum_{i=1}^{n} m_i \|u_i\|^2 \geq \frac{\min\{m_i, 1 \leq i \leq n\}}{2} \sum_{i=1}^{n} \|u_i\|^2 \quad \text{for all } u \in X. \tag{3.3}$$

Furthermore, form Assumption (iii), there exist two positive constants σ and κ satisfying $\sigma < \frac{1}{\tau}$ and

$$\frac{(b-a)^2}{2\min\{m_i, 1 \leq i \leq n\}} F(x, t_1, \ldots, t_n) \leq \sigma \sum_{i=1}^{n} t_i^2 + \kappa$$

for all $(t_1, \ldots, t_n) \in \mathbb{R}^n$ uniformly with respect to $x \in [a,b]$. Fix $u = (u_1, \ldots, u_n) \in X$, so

$$F(x, u_1(x), \ldots, u_n(x)) \leq \frac{2\min\{m_i, 1 \leq i \leq n\}}{(b-a)^2} \left(\sigma \sum_{i=1}^{n} |u_i(x)|^2 + \kappa\right) \quad \text{for a.e. } x \in [a,b]. \tag{3.4}$$

Moreover, since

$$\max_{x \in [a,b]} |u_i(x)| \leq \frac{(b-a)^{\frac{1}{2}}}{2} \|u_i\| \quad \text{for all } u_i \in W_0^{1,2}([a,b])$$

for $1 \leq i \leq n$, we have

$$\sup_{x \in [a,b]} \sum_{i=1}^{n} |u_i(x)|^2 \leq \frac{b-a}{4} \sum_{i=1}^{n} \|u_i\|^2 \tag{3.5}$$

for each $u = (u_1, \dots, u_n) \in X$. Then, for any fixed $\lambda \in]0, \tau[$, from (3.1)-(3.5) we obtain

$$\begin{aligned}
\Phi(u) + \lambda J(u) &= \frac{1}{2}\sum_{i=1}^{n} \tilde{K}_i(||u_i||^2) - \lambda \int_a^b F(x, u_1(x), \dots, u_n(x))dx \\
&\geq \frac{1}{2}\sum_{i=1}^{n} \tilde{K}_i(||u_i||^2) - \frac{2\sigma\lambda \min\{m_i, 1 \leq i \leq n\}}{(b-a)^2} \sum_{i=1}^{n} \int_a^b |u_i(x)|^2 dx \\
&\quad - \frac{2\lambda\kappa \min\{m_i, 1 \leq i \leq n\}}{b-a} \\
&\geq \frac{\min\{m_i, 1 \leq i \leq n\}}{2} \sum_{i=1}^{n} \|u_i\|^2 - \frac{2\sigma\tau \min\{m_i, 1 \leq i \leq n\}}{(b-a)^2} \sum_{i=1}^{n} \int_a^b |u_i(x)|^2 dx \\
&\quad - \frac{2\tau\kappa \min\{m_i, \ \ 1 \leq i \leq n\}}{b-a} \\
&\geq \frac{\min\{m_i, 1 \leq i \leq n\}}{2} \sum_{i=1}^{n} \|u_i\|^2 - \frac{\sigma\tau \min\{m_i, 1 \leq i \leq n\}}{2} \sum_{i=1}^{n} \|u_i\|^2 \\
&\quad - \frac{2\tau\kappa \min\{m_i, 1 \leq i \leq n\}}{b-a} \\
&\geq \frac{\min\{m_i, 1 \leq i \leq n\}}{2}(1 - \sigma\tau) \sum_{i=1}^{n} \|u_i\|^2 - \frac{2\tau\kappa \min\{m_i, 1 \leq i \leq n\}}{b-a}.
\end{aligned}$$

So, for each $\lambda \in \,]0, \tau[$, we have

$$\lim_{||u|| \to +\infty} (\Phi(u) + \lambda J(u)) = +\infty.$$

To check the other assumptions in Theorem 1.1 we use Proposition 1.2. To this end, we see that the required hypothesis $\Phi(u_1) > r$ follows from (i) and (3.1) by choosing $u_1 = w$. Also, by choosing $u_0 = 0$, we see that $\Phi(u_0) = 0$, and since $F(x, 0, \dots, 0) = 0$ for every $x \in [a, b]$, from (3.2) we have $J(u_0) = 0$. Taking (3.5) into account, from (3.3) for each $r > 0$ we get

$$\begin{aligned}
\Phi^{-1}(]-\infty, r[) &= \{u = (u_1, u_2, \dots, u_n) \in X; \Phi(u) < r\} \\
&= \left\{ u \in X; \frac{\min\{m_i, 1 \leq i \leq n\}}{2} \sum_{i=1}^{n} \|u_i\|^2 < r \right\} \\
&\subseteq \left\{ u \in X; \sum_{i=1}^{n} |u_i(x)|^2 \leq \frac{r(b-a)}{2\min\{m_i, 1 \leq i \leq n\}} \ \text{ for each } x \in [a, b] \right\}.
\end{aligned}$$

Now, since $M_1 > 0$ we have

$$\int_a^b \sup_{(t_1, \dots, t_n) \in A_1} F(x, t_1, \dots, t_n) dx < 2r \frac{\int_a^b F(x, w(x))dx}{\sum_{i=1}^{n} \tilde{K}_i(\int_a^b |w_i'(x)|^2 dx)},$$

and we obtain

$$\begin{aligned}\sup_{u\in\Phi^{-1}(]-\infty,r[)}(-J(u)) &= \sup_{\frac{\min\{m_i,\ 1\le i\le n\}}{2}\sum_{i=1}^{n}||u_i||^2<r}\int_a^b F(x,u(x))dx \\ &\le \int_a^b \sup_{(t_1,\dots,t_n)\in A_1} F(x,t_1,\dots,t_n)dx \\ &< 2r\frac{\int_a^b F(x,w(x))dx}{\sum_{i=1}^{n}\tilde{K}_i(\int_a^b |w_i'(x)|^2dx)} \\ &= r\frac{-J(w(x))}{\Phi(w(x))},\end{aligned}$$

so

$$\sup_{u\in\Phi^{-1}(]-\infty,r[)}(-J(u)) < r\frac{-J(w)}{\Phi(w)},$$

which is another required assumption in Theorem 1.1. Fix $1 < h < \frac{\tau M_1}{r}$ and ρ such that

$$\sup_{u\in\Phi^{-1}(]-\infty,r[)}(-J(u)) + \frac{r\frac{-J(w)}{\Phi(w)} - \sup_{\Phi(u)<r}(-J(u))}{h} < \rho < r\frac{-J(w)}{\Phi(w)},$$

due to $\tau > \frac{r}{M_1}$, from Proposition 1.2, with $u_0 = 0$ and $u_1 = w$ we obtain

$$\sup_{\lambda\in R}\inf_{u\in X}(\Phi(u)+\lambda(J(u)+\rho)) < \inf_{u\in X}\sup_{\lambda\in[0,\tau]}(\Phi(u)+\lambda(J(u)+\rho)).$$

Moreover, since $G : [a,b]\times\mathbb{R}^n \to \mathbb{R}$ is a measurable function with respect to x in $[a,b]$ for every $(t_1,\dots,t_n)\in\mathbb{R}^n$, and is a C^1-function with respect to $(t_1,\dots,t_n)\in\mathbb{R}^n$ for every x in $[a,b]$ satisfying the condition (1.2), then the functional

$$\Psi(u) = -\int_a^b G(x,u_1(x),\dots,u_n(x))dx$$

is well defined and continuously Gâteaux differentiable on X, with compact derivative, and one has

$$\Psi'(u)(v) = -\int_a^b\sum_{i=1}^{n} G_{u_i}(x,u_1(x),\dots,u_n(x))v_i(x)dx$$

for all $v\in X$. Now, all the assumptions of Theorem 1.1 are satisfied. Hence, taking into account that the critical points of the functional $\Phi+\lambda J+\mu\Psi$ are exactly the weak solutions of the system (1.1), we have the conclusion.

Acknowledgements. Research by Shapour Heidarkhani was in part supported by a grant from IPM(No. 89350020).

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