BOUNDARY VALUE PROBLEMS FOR DYNAMIC EQUATIONS WITH ADVANCED ARGUMENTS ON TIME SCALES

TADEUSZ JANKOWSKI

Department of Differential Equations, Gdansk University of Technology
11/12 G.Narutowicz Str., 80-952 Gdansk, Poland.

Dedicated to Professor V.Lakshmikantham on his 85th birthday

ABSTRACT. This paper considers boundary value problems on time scales and also discusses inequalities on time scales. We formulate sufficient conditions under which such problems have extremal solutions in a corresponding region bounded by upper and lower solutions. Examples are also included to illustrate the importance of the result obtained.

AMS (MOS) Subject Classification. 34A10, 34A45

1. INTRODUCTION

Stefan Hilger [4] introduced the calculus of measure chains on order to unify continuous and discrete analysis. Major works devoted to the calculus on time scales have been introduced in papers [1, 3, 5, 10].

Throughout this paper, we denote by $\mathbb{T}$ any time scale (nonempty closed subset of the real numbers $\mathbb{R}$). We assume that $0, T \in \mathbb{T}$ and denote by $J = [0, T]$ a subset of $\mathbb{T}$ such that $[0, T] = \{t \in \mathbb{T} : 0 \leq t \leq T\}$. By σ we denote the forward jump operator $\sigma(t) = \inf\{s \in \mathbb{T} : s > t\}$. The graininess function $\mu : \mathbb{T} \to \mathbb{R}_+$ is defined by $\mu(t) = \sigma(t) - t$ with $\mathbb{R}_+ = [0, \infty)$. Let $C(J, \mathbb{R})$ denote the set of continuous functions $u : J \to \mathbb{R}$.

In this paper, we investigate the following first order dynamic equation on time scales of the form

$$(1.1) \qquad \begin{cases} x^{\triangle}(t) &= f(t, x(t), x(\alpha(t))) \equiv (\mathcal{F}x)(t), \quad t \in [0, T], \\ 0 &= g(x(0), x(T)), \end{cases}$$

where $f \in C(J \times \mathbb{R} \times \mathbb{R}, \mathbb{R})$, $\alpha \in C(J, J)$, $g \in C(\mathbb{R} \times \mathbb{R}, \mathbb{R})$.

Differential and difference equations are special cases of dynamic equations. To find solutions of nonlinear differential equations both with initial or boundary conditions we can use the monotone iterative method based on lower and upper solutions. This technique is well known and we have many applications. Recently, it is also applied to differential equations with deviating arguments (delayed or advanced). This method is also used for dynamic ones (see, for example [2, 3, 6, 7, 11, 12] but it is not so extensive as for continuous case. Dynamic equations with deviating arguments are discussed, for example in papers [6, 9]. This paper extends the application on problems of type (1.1). We prove the existence of minimal and maximal solutions for problems (1.1) by using the Heikkila and Lakshmikantham theorem [4]. It is important to indicate that the right–hand–side of our problem depends on a solution x at an advanced argument α.

The plan of this paper is as follows. In Section 2, we discussed dynamic inequations with advanced arguments. In Section 3, we formulate sufficient conditions when advanced nonlinear dynamic equation with an initial condition at the end point T has a unique solution. A linear case is also discussed. The existence of solutions for dynamic problems of type (1.1) is discussed in Section 4. The last Section 5 contains dynamic problems having more advanced arguments α. Some examples are added to illustrate theoretical results.

2. DYNAMIC INEQUALITIES

In this section we present some linear dynamic inequalities which are needed in Section 4.

Lemma 2.1. *Assume that* $m \in C(J, \mathbb{R})$, $1 + \sup_{t \in J} \mu(t)m(t) > 0$ *and let*

$$\begin{cases} x^{\triangle}(t) & \geq & m(t)x(t), \quad t \in [0,T], \\ x(T) & \leq & 0. \end{cases} \tag{2.1}$$

Then $x(t) \leq 0$, $t \in J$.

Proof. We replace the inequality in (2.1) by

$$x^{\triangle}(t) = m(t)x(t) + Q(t), \quad t \in J$$

with $Q \in C(J, \mathbb{R}_+)$. It yields

$$x(t) = e_m(t,0)\left[x(0) + \int_0^t e_m(0,\sigma(s))Q(s)\Delta s\right], \tag{2.2}$$

by Theorem 2.77 [3]. For a discussion of the exponential function e, see for example Section 2.2 of [3]. Take $t = T$ and find $x(0)$ to obtain

$$x(0) = e_m(0,T)x(T) - \int_0^T e_m(0,\sigma(s))Q(s)\Delta s.$$

Substituting the last relation to formula (2.2), we get

$$x(t) = e_m(t,T)x(T) - \int_t^T e_m(t,\sigma(s))Q(s)\Delta s \leq 0$$

because $e_m(t,\sigma(s)) > 0$ (by Theorem 2.48(ii) of [3], $Q(s) \geq 0$, $s \in J$ and $x(T) \leq 0$. This ends the proof. □

Remark 2.2. Let $m(t) = 0$, $t \in J$. If

$$\begin{cases} x^{\triangle}(t) & \geq & 0, \quad t \in [0,T], \\ x(T) & \leq & 0, \end{cases}$$

then $x(t) \leq 0$, $t \in J$, by Lemma (2.1)

Lemma 2.3. *Assume that*

H_1 : *there exist functions* $n \in C(J, \mathbb{R}_+)$, $\alpha \in C(J, J)$, $t \leq \alpha(t) \leq T$ *and* $\alpha(t) \not\equiv t$ *on* J.

In addition, we assume that

$$\rho_1 \equiv \int_0^T n(t)\Delta t \leq 1.$$

Let

$$\begin{cases} x^{\triangle}(t) & \geq & n(t)x(\alpha(t)), \quad t \in [0,T], \\ x(T) & \leq & 0. \end{cases} \tag{2.3}$$

Then $x(t) \leq 0$, $t \in J$.

Proof. We need to prove that $x(t) \leq 0$, $t \in J$. Suppose that the inequality $x(t) \leq 0$, $t \in J$ is not true. Then, we can find $t_0 \in [0,T)$ such that $x(t_0) > 0$. Put

$$x(t_1) = \min_{[t_0,T]} x(t) \leq 0.$$

Integrating the dynamic inequality in (2.3) from t_0 to t_1, we obtain

$$\begin{aligned} x(t_1) - x(t_0) &\geq \int_{t_0}^{t_1} n(t)x(\alpha(t))\Delta t \\ &\geq x(t_1)\int_0^T n(t)\Delta t \geq x(t_1). \end{aligned}$$

It contradicts the assumption that $x(t_0) > 0$. This proves that $x(t) \leq 0$ on J and the proof is complete. □

Lemma 2.4. *Assume that Assumptions* H_1, H_2 *hold with*

H_2 : *there exists a continuous function* $m : J \to \mathbb{R}$ *such that*

$$\sup_{t \in J}[\mu(t)m(t)] > -1.$$

In addition, we assume that

$$\rho_2 \equiv \int_0^T \mathcal{N}(t)\Delta t \leq 1 \quad \textit{with} \quad \mathcal{N}(t) = \frac{n(t)}{1 + \mu(t)m(t)}\, e_m(\alpha(t), t). \tag{2.4}$$

Let

$$\begin{cases} x^{\triangle}(t) & \geq & m(t)x(t) + n(t)x(\alpha(t)), \quad t \in [0,T], \\ x(T) & \leq & 0. \end{cases} \tag{2.5}$$

Then $x(t) \leq 0$, $t \in J$.

Proof. Let $p(t) = e_{\Theta m}(t,0)x(t)$ with $\Theta m = -\frac{m(t)}{1+\mu(t)m(t)}$. Then, we have

$$\begin{aligned} p^{\triangle}(t) &= \Theta m\, e_{\Theta m}(t,0)x(t) + e_{\Theta m}(\sigma(t),0)x^{\triangle}(t) \\ &= -\frac{m(t)}{1+\mu(t)m(t)}\, e_{\Theta m}(t,0)x(t) + \left[1 - \frac{\mu(t)m(t)}{1+\mu(t)m(t)}\right] e_{\Theta m}(t,0)x^{\triangle}(t) \\ &= \frac{1}{1+\mu(t)m(t)}\, e_{\Theta m}(t,0)\left[-m(t)x(t) + x^{\triangle}(t)\right] \\ &\geq \frac{1}{1+\mu(t)m(t)}\, e_{\Theta m}(t,0)n(t)x(\alpha(t)) \\ &= \frac{1}{1+\mu(t)m(t)}\, e_{\Theta m}(t,0)\frac{1}{e_{\Theta m}(\alpha(t),0)}n(t)x(\alpha(t)) \\ &= \mathcal{N}(t)p(\alpha(t)), \end{aligned}$$

Then problem (2.5) takes the form

$$\begin{cases} p^{\triangle}(t) & \geq & \mathcal{N}(t)p(\alpha(t)), \quad t \in J, \\ p(T) & \leq & 0. \end{cases}$$

It yields $p(t) \leq 0$ on J, by Lemma (2.1). It shows that $x(t) \leq 0$ on J. The proof is complete. □

Remark 2.5. If $m(t) \equiv 0$, then $e_m(s,t) \equiv 1$, by Theorem 2.36(i) of [3]. Then Lemma 2.4 reduces to Lemma 2.3.

Remark 2.6. If $\mathbb{T} = \mathbb{R}$, then $\mu(t) = 0$ and $e_m(\alpha(t),t) = \exp\left(\int_t^{\alpha(t)} m(s)ds\right)$. In this case, ρ_2 from condition (2.4) has the form

$$\rho_2 = \int_0^T n(t)e^{\int_t^{\alpha(t)} m(s)ds}dt$$

and in this case Lemma (2.4) reduces to Lemma 1 of [8].

Remark 2.7. Assume that $m(t) \geq 0$ on J. Then $e_m(\alpha(t),t) \leq e_m(T,t)$. If we assume that

$$\rho_3 \equiv \int_0^T \mathcal{N}_1(t)\Delta t \leq 1 \quad \text{with} \quad \mathcal{N}_1(t) = \frac{n(t)}{1+\mu(t)m(t)}\, e_m(T,t)$$

then condition (2.4) holds. Note that ρ_3 does not depend on α.

Remark 2.8. Let $\mathbb{T} = \mathbb{R}$ and $m(t) = m > 0$, $n(t) = n > 0$. Then ρ_3 from Remark 2.7 takes the form

$$\rho_3 = \frac{n}{m}\left(e^{mT} - 1\right).$$

Example 2.9. Let $\mathbb{T} = Z^+$, $J = \{0 \leq j \leq L,\, j \in Z^+\}$ and

$$\begin{cases} x(n+1) & \geq & (1+c)x(n) + dx(\alpha(n)), \quad n = 0,1,\ldots,L-1, \\ x(L) & \leq & 0, \end{cases} \tag{2.6}$$

where $c > -1$, $d \geq 0$ and $\alpha(n)$ is a fixed number such that $\alpha(n) \in \{n, n+1, \ldots, L\}$ and $\alpha(P) \not\equiv P$ on the set $P = \{0, 1, \ldots, L\}$. In this case $\mu(t) = 1$, $t \in J$ and Assumption H_2 holds because $\sup_{t\in J}[\mu(t)m(t)] = c > -1$.

If we assume that

$$\frac{d}{c(1+c)}\left[(1+c)^L - 1\right] \leq 1,$$

then any solution x of problem (2.6) satisfies the relation $x(n) \leq 0$, $n \in P$, by Lemma 2.4 and Remark 2.7.

3. APPLICATION OF BANACH FIXED POINT THEOREM

Consider the following problem

$$x^{\triangle}(t) = (\mathcal{F}x)(t), \quad t \in J, \quad x(T) = k_0 \in \mathbb{R}, \tag{3.1}$$

where operator $\mathcal{F}$ is defined as in problem (1.1). In the next theorem we formulate sufficient conditions under which problem (3.1) has a unique solution. To do it we apply the Banach fixed point theorem.

Theorem 3.1. *Suppose that*

$H_3:$ $f \in C(J \times \mathbb{R} \times \mathbb{R}, \mathbb{R})$, $\alpha \in C(J, J)$, $t \leq \alpha(t) \leq T$ *on* J,
$H_4:$ *there exist nonnegative constants* L_1, L_2 *such that*

$$|f(t, x_1, x_2) - f(t, \bar{x}_1, \bar{x}_2)| \leq L_1|x_1 - \bar{x}_1| + L_2|x_2 - \bar{x}_2|$$

for $t \in J$, $x_1, x_2, \bar{x}_1, \bar{x}_2 \in \mathbb{R}$.

Then problem (3.1) *has a unique solution* $x \in C^1(J, \mathbb{R})$.

Proof. Integrating (3.1), we have

$$x(t) = k_0 - \int_t^T (\mathcal{F}x)(s)\Delta s \equiv (Ax)(t), \quad t \in J.$$

It means that solving (3.1)) is equivalent to solving a fixed point problem of operator A. Now we use the Banach fixed point theorem. Let

$$X = \{x \in C_{rd}(J, \mathbb{R}) \quad \text{with} \quad \|x\| = \max_{t\in J} e_\lambda(t, T)|x(t)|\}$$

with a constant $\lambda > 0$ such that $\lambda \geq L_1 + L_2$. Note X is a Banach space. For $u, v \in X$, we have

$$\begin{aligned}\|Au - Av\| &\leq \max_{t\in J} e_\lambda(t, T) \int_t^T |(\mathcal{F}u)(s) - (\mathcal{F}v)(s)|\Delta s \\ &\leq \max_{t\in J} e_\lambda(t, T) \int_t^T \left[L_1|u(s) - v(s)| + L_2|u(\alpha(s)) - v(\alpha(s))|\right]\Delta s \\ &= \max_{t\in J} e_\lambda(t, T) \int_t^T \left[L_1 e_\lambda(T, s) e_\lambda(s, T)|u(s) - v(s)|\right.\end{aligned}$$

$$+L_2 e_\lambda(T,\alpha(s))e_\lambda(\alpha(s),T)|u(\alpha(s))-v(\alpha(s))|]\,\Delta s$$
$$\leq \|u-v\|H,$$

where

$$H = \max_{t\in J} e_\lambda(t,0)e_\lambda(0,T)\int_t^T [L_1 e_\lambda(T,0)e_\lambda(0,s)+L_2 e_\lambda(T,0)e_\lambda(0,\alpha(s))]\Delta s$$
$$= \max_{t\in J} e_\lambda(t,0)\int_t^T [L_1 e_\lambda(0,s)+L_2 e_\lambda(0,\alpha(s))]\Delta s.$$

Indeed, $e_\lambda(0,\alpha(t)) \leq e_\lambda(0,t)$, $t\in J$, by Definition 2.30 of [3]. Hence

$$H \leq (L_1+L_2)\max_{t\in J} e_\lambda(t,0)\int_t^T e_\lambda(0,s)\Delta s.$$

Now, also

$$\int_t^T e_\lambda(0,s)\Delta s = \int_t^T e_{\Theta\lambda}(s,0)\Delta s = \int_t^T \frac{e^{\triangle}_{\Theta\lambda}(s,0)}{\Theta\lambda}\Delta s \leq \frac{1}{\lambda}\left[e_{\Theta\lambda}(t,0)-e_{\Theta\lambda}(T,0)\right]$$
$$= \frac{1}{\lambda}\left[e_\lambda(0,t)-e_\lambda(0,T)\right]$$

because

$$\Theta\lambda = -\frac{\lambda}{1+\mu(t)\lambda} \geq -\lambda.$$

It yields

$$H \leq \frac{L_1+L_2}{\lambda}\max_{t\in J} e_\lambda(t,0)[e_\lambda(0,t)-e_\lambda(0,T)] = \frac{L_1+L_2}{\lambda}[1-e_\lambda(0,T)]$$
$$\leq 1-e_\lambda(0,T) \equiv \xi < 1.$$

As a result

$$\|Au-Av\| \leq \|u-v\|\xi.$$

In view of the Banach fixed point theorem, problem (3.1) has a unique solution. This ends the proof. □

Remark 3.2. If $\mathbb{T}=\mathbb{R}$, then

$$e_\lambda(t,T)=e^{\lambda(t-T)},\quad \lambda\geq L_1+L_2,\ \ \lambda>0\quad\text{and}\quad \|x\|=\max_{t\in J} e^{\lambda(t-T)}|x(t)|.$$

Remark 3.3. Now we consider the linear dynamic equation of the form

$$\begin{cases} x^{\triangle}(t) &= (\mathcal{L}x)(t)+h(t),\quad t\in J,\\ x(T) &= \bar{h}\in\mathbb{R}, \end{cases} \tag{3.2}$$

where operator $\mathcal{L}$ is defined by

$$(\mathcal{L}x)(t)=m(t)x(t)+n(t)x(\alpha(t)) \tag{3.3}$$

with $m, h \in C(J, \mathbb{R})$, $n \in C(J, \mathbb{R}_+)$, $\alpha \in C(J, J)$ and $t \leq \alpha(t) \leq T$ on J. In this case, solving (3.2) is equivalent to solving the following problem

$$x(t) = \bar{h} - \int_t^T [(\mathcal{L}x)(s) + h(s)]\Delta s \equiv (\mathcal{A}_h x)(t). \tag{3.4}$$

Then, in view of Theorem (3.1), problem (3.2) has a unique solution.

We can also replace problem (3.2)) in another way. Using Theorem 2.77 of [3] to the dynamic equation (3.2) we obtain

$$x(t) = e_m(t,0)\left\{x(0) + \int_0^t e_m(0,\sigma(s))[n(s)x(\alpha(s)) + h(s)]\Delta s\right\}.$$

Now, using the condition $x(T) = \bar{h}$, we finally have

$$x(t) = e_m(t,0)\left\{e_m(0,T)\bar{h} - \int_t^T e_m(0,\sigma(s))[n(s)x(\alpha(s)) + h(s)]\Delta s\right\}.$$

4. EXISTENCE OF SOLUTIONS OF PROBLEM (1.1)

Now, we derive a fixed point result for nondecreasing mappings in ordered spaces which play a central role in our investigations. We say that $Q : [a,b] \to [a,b]$ is nondecreasing if $Qx \leq Qy$ for $x, y \in [a,b]$ and $x \leq y$. We say that $x \in [a,b]$ is the least fixed point of Q in $[a,b]$ if $x = Qx$ and if $x \leq y$ whenever $y \in [a,b]$ and $y = Qy$. The greatest fixed point of Q in $[a,b]$ is defined similarly, by reversing the inequality. If both least and greatest fixed point of Q in $[a,b]$ exist, we call them extremal fixed points of Q in $[a.b]$.

Theorem 4.1 ([4]). *Let $[a,b]$ be an ordered interval in a subset Y of an ordered Banach space X and let $Q : [a,b] \to [a,b]$ be a nondecreasing mapping. If each sequence $\{Qx_n\} \subset Q([a,b])$ converges, whenever $\{x_n\}$ is a monotone sequence in $[a,b]$, then the sequence of Q–iteration of a converges to the least fixed point x_* of Q and the sequence of Q–iteration of b converges to the greatest fixed point x^* of Q. Moreover,*

$$x_* = \min\{y \in [a,b] : y \geq Qy\}, \quad \text{and} \quad x^* = \max\{y \in [a,b] : y \leq Qy\}.$$

Let us introduce the following definitions.

Definition 4.2. A function $x_0 \in C^1(J, \mathbb{R})$ is said to be a lower solution of (1.1)) if

$$x_0^{\triangle}(t) \leq (\mathcal{F}x_0)(t), \quad g(x_0(0), x_0(T)) \leq 0.$$

Definition 4.3. A function $y_0 \in C^1(J, \mathbb{R})$ is said to be an upper solution of (1.1) if

$$y_0^{\triangle}(t) \geq (\mathcal{F}y_0)(t), \quad g(y_0(0), y_0(T)) \geq 0.$$

Now we formulate the main result of this paper.

Theorem 4.4. *Assume that* $f \in C(J \times \mathbb{R} \times \mathbb{R}, \mathbb{R})$, $g \in C(\mathbb{R} \times \mathbb{R}, \mathbb{R})$. *Let* $x_0, y_0 \in C^1(J, \mathbb{R})$ *be lower and upper solutions of* (1.1), *respectively and* $y_0(t) \le x_0(t)$, $t \in J$. *Moreover, we assume that there exists functions* m, n *such that both Assumptions* H_1, H_2 *and condition* (2.4) *hold and*

$$f(t, u_1, u_2) - f(t, v_1, v_2) \ge -m(t)[v_1 - u_1] - n(t)[v_2 - u_2] \tag{4.1}$$

if $y_0(t) \le u_1 \le v_1 \le x_0(t)$, $y_0(\alpha(t)) \le u_2 \le v_2 \le x_0(\alpha(t))$. *In addition, we assume that* g *is nondecreasing in the first variable and there exists a constant* $M > 0$ *such that*

$$g(v, u) - g(v, \bar{u}) \le M(\bar{u} - u) \quad \text{if } y_0(T) \le u \le \bar{u} \le x_0(T).$$

Then problem (1.1) *has minimal and maximal solutions in the region* $[y_0, x_0] = \{u \in C(J, \mathbb{R}) : y_0(t) \le u(t) \le x_0(t),\ t \in J\}$.

Proof. Assume that operator F is nonincreasing in region $[y_0, x_0]$ and also let G_h be nondecreasing with respect to h. Choose $h_1, h_2 \in C(J, \mathbb{R})$ such that $h_1(t) \le h_2(t)$ on J. Let x_1, x_2 denote the solutions of problem (3.2) with Fh_1, Fh_2 instead of h, and with G_{h_1}, G_{h_2} instead of $\bar{h}$, respectively. Since problem (3.2) has a unique solution for each $h \in C(J, \mathbb{R})$, $\bar{h} \in \mathbb{R}$, then x_1, x_2 are well defined. Put $x = x_1 - x_2$. Then,

$$x^{\triangle}(t) = (\mathcal{L}x)(t) + (Fh_1)(t) - (Fh_2)(t) \ge (\mathcal{L}x)(t), \quad t \in J,$$
$$p(T) = G_{h_1} - G_{h_2} \le 0.$$

In view of Lemma (2.4), we see that $x_1(t) \le x_2(t)$ on J, so the operator $\mathcal{A}_h$ is nondecreasing. It is also continuous.

For $u \in [y_0, x_0]$, we put

$$Fu = \mathcal{F}u - \mathcal{L}u, \quad G_u = \frac{1}{M} g(u(0), u(T)) + u(T),$$

where the operator $\mathcal{F}$ is defined as in problem (1.1). It is easy to see that operator F is nonincreasing in $[y_0, x_0]$. Moreover, G_u is nondecreasing with respect to u. We define the operator $A = \mathcal{A}_F$. Let $x_1 = Ay_0$, $x_2 = Ax_0$, so

$$\begin{cases} x_1^{\triangle}(t) &= (\mathcal{L}x_1)(t) + (Fy_0)(t), \\ x_1(T) &= G_{y_0}, \end{cases}$$

and

$$\begin{cases} x_2^{\triangle}(t) &= (\mathcal{L}x_2)(t) + (Fx_0)(t), \\ x_2(T) &= G_{x_0}. \end{cases}$$

Now apply Lemma (2.4) with $x(t) = x_2(t) - x_0(t)$ and its easy to show using the definition of the lower solution x_0 that $x_0(t) \ge x_2(t) = (A\,x_0)(t)$. Similarly we can show $(A\,y_0)(t) = x_1(t) \ge y_0(t)$ on J. Put $x(t) = x_1(t) - x_2(t)$. Then

$$x^{\triangle}(t) = (\mathcal{L}x_1)(t) + (Fy_0)(t) - (\mathcal{L}x_2)(t) - (Fx_0)(t) \ge (\mathcal{L}x)(t),$$
$$x(T) = G_{y_0} - G_{x_0} \le 0$$

. Using again Lemma (2.4), we see that $x_1(t) \leq x_2(t)$ on J, so the operator A is nondecreasing. It means that $y_0 \leq Au \leq x_0$ for $u \in [y_0, x_0]$. Hence $A : [y_0, x_0] \to [y_0, x_0]$ and operator A is bounded because $\|Au\| \leq \max(\|y_0\|, \|x_0\|)$.

Let $\{y_n\}$ be a monotone sequence in $[y_0, x_0]$, so $y_0 \leq Ay_n \leq x_0$. Hence $\|Ay_n\| \leq K$. It is easy to show that $\{Ay_n\}$ is equicontinuous. By Arzeli–Ascoli theorem, $\{Ay_n\}$ is relative compact. It proves that $\{Ay_n\}$ converges in $A([y_0, x_0])$. Finally, operator A has a least and a greatest fixed point in $[y_0, x_0]$, by Theorem (4.1). It results that problem (1.1) has minimal and maximal solutions in $[y_0, x_0]$. This ends the proof. □

Remark 4.5. If we assume that f is nonincreasing with respect to the last variable, the condition (4.1) holds with $n(t) = 0$, $t \in J$. Note that, in this case, condition (2.4) holds too.

Example 4.6. (see [8]). Let $\mathbb{T} = \mathbb{R}$. Consider the problem

$$(4.2) \quad \begin{cases} x'(t) &= 2e^{x(t)} + (\sin t)e^{-2e(\sqrt{t}-t)}x(\sqrt{t}) - 1 \equiv (\mathcal{F}x)(t), \quad t \in J = [0,1], \\ 0 &= x(0) + x^2(0) - x(1). \end{cases}$$

Note that $\alpha(t) = \sqrt{t}$, and $t \leq \alpha(t) \leq T = 1$. Put $x_0(t) = t$, $y_0(t) = -1$, $t \in J$. It yields

$$(\mathcal{F}x_0)(t) = 2e^t + (\sin t)e^{-2e(\sqrt{t}-t)}\sqrt{t} - 1 \geq 1 = x_0'(t),$$
$$(\mathcal{F}y_0)(t) = 2e^{-1} - (\sin t)e^{-2e(\sqrt{t}-t)} - 1 < 0 = y_0'(t),$$

and

$$g(x_0(0), x_0(1)) = g(0,1) = -1 < 0, \quad g(y_0(0), y_0(1)) = g(-1,-1) = 1 > 0.$$

It proves that x_0, y_0 are lower and upper solutions of problem (4.2), respectively. Indeed, $m(t) = 2e^t$, $n(t) = (\sin t)e^{-2(\sqrt{t}-t)e}$, $M = 1$. Moreover,

$$\int_0^1 n(t)e^{\int_t^{\alpha(t)} m(s)ds}dt \leq \int_0^1 \sin t\, dt = 1 - \cos 1 < 1,$$

so condition (2.4) holds too. By Theorem (4.4), problem (4.2) has extremal solutions in the region $[-1, t]$.

Example 4.7. Let $\mathbb{T} = Z^+$, $J = \{0 \leq j \leq L,\ j \in Z^+\}$, so $x^{\triangle}(i) = \Delta x(i) = x(i+1) - x(i)$. We consider the problem

$$(4.3) \quad \begin{cases} \Delta x(i) &= b|\sin i|x(i) - |\sin i|x(\alpha(i)) \equiv (\mathcal{F}x)(i), \quad i = 0, 1, \ldots, L-1, \\ 0 &= \lambda[x(0) + x^2(0)] - \beta x(L) + \gamma, \end{cases}$$

where $b \geq 1$, $\beta > 0$, $\lambda > 0$ and $\alpha(i)$ is a fixed number such that $\alpha(i) \in \{i, i+1, \ldots, L\}$ and $\alpha(P) \not\equiv P$ on the set $P = \{0, 1, \ldots, L\}$. In this case $\mu(t) = 1$, $t \in J$, so Assumption H_2 holds.

We assume that

$$\gamma \le 0, \quad \beta a + \gamma \ge 0. \tag{4.4}$$

Put $x_0(i) = 0$, $y_0(i) = -a$, $i \in P$ for $a \ge 1$. Then

$$(\mathcal{F}x_0)(i) = 0 = \Delta x_0(i),$$
$$(\mathcal{F}y_0)(i) = (1-b)a|\sin i| \le 0 = \Delta y_0(i),$$

and

$$g(x_0(0), x_0(L)) = g(0,0) = \gamma \le 0,$$
$$g(y_0(0), y_0(L)) = g(-a,-a) = \lambda a(a-1) + \beta a + \gamma \ge 0,$$

by (4.4). Functions x_0, y_0 are lower and upper solutions of problem (4.3), respectively. It is easy to see that $m(i) = b|\sin i|$, $n(i) = 0$, $t \in J$ and $M = \beta$. In view of Theorem 4.4, problem (4.3) has extremal solutions.

5. GENERALIZATIONS

In this section we consider a boundary value problem of the form

$$\begin{cases} x^{\triangle}(t) &= f(t, x(t), x(\alpha_1(t)), \ldots, x(\alpha_r(t))) \equiv (\mathcal{G}x)(t), \quad t \in J = [0,T], \\ 0 &= g(x(0), x(T)). \end{cases} \tag{5.1}$$

We formulate only corresponding results using the notions of lower and upper solutions of problem (5.1) which are the same as before with the operator $\mathcal{G}$ instead of operator $\mathcal{F}$. The next theorem is similar to Theorem (4.4) and therefore the proof is omitted.

Theorem 5.1. *Assume that* $f \in C(J \times \mathbb{R}^{r+1}, \mathbb{R})$, $g \in C(\mathbb{R} \times \mathbb{R}, \mathbb{R})$, $\alpha_i \in C(J,J)$, $t \le \alpha_i(t) \le T$ *and* $\alpha_i(t) \not\equiv t$ *on* J *for* $i = 1,2,\ldots,r$. *Let* $x_0, y_0 \in C^1(J,\mathbb{R})$ *be lower and upper solutions of* (5.1), *respectively and* $y_0(t) \le x_0(t)$, $t \in J$. *We assume that there exists functions* $m \in C(J,\mathbb{R})$, $n_i \in C(J,\mathbb{R}_+)$, $i = 1,2,\ldots r$ *such that*

$$f(t,u_0,u_1,\ldots,u_r) - f(t,v_0,v_1,\ldots,v_r) \ge -m(t)[v_0 - u_0] - \sum_{i=1}^{r} n_i(t)[v_i - u_i]$$

if $t \in J$, $y_0(\alpha_i(t)) \le u_i \le v_i \le x_0(\alpha_i(t))$, $i = 0,1,\ldots,r$ *with* $\alpha_0(t) = t$. *Moreover, we assume that* $\sup_{t\in J}[\mu(t)m(t)] > -1$ *and condition* (2.4) *holds with*

$$\mathcal{N}(t) = \sum_{i=1}^{r} \frac{n_i(t)}{1+\mu(t)m(t)}\, e_m(\alpha(t),t).$$

In addition, we assume that g is nondecreasing in the first variable and there exists a constant $M > 0$ such that

$$g(v,u) - g(v,\bar{u}) \le M(\bar{u} - u) \quad \textit{if } y_0(T) \le u \le \bar{u} \le x_0(T).$$

Then problem (5.1) *has minimal and maximal solutions in the region* $[y_0, x_0]$.

REFERENCES

[1] R.P. Agarwal and M. Bohner, Basic calculus on time scales and some of its applications, *Results Math.*, 35:3–22, 1999.

[2] R.P. Agarwal, T. Jankowski and D.O' Regan, Dynamic inequalities and equations of Volterra type, *Dynam. Contin. Discrete Impuls. Syst. Ser. A Math. Anal.,* 16:471–480, 2009.

[3] M. Bohner and A. Peterson, Dynamic Equations on Time Scales, Birkhauser, Boston 2001.

[4] S. Heikkila and V. Lakshmikantham, Monotone Iterative Technique for Discontinuous Nonlinear Differential Equations, Dekker, New York, 1994.

[5] S. Hilger, Analysis on measure chains - A unified approach to continuous and discrete calculus, *Results Math.,* 18:18–56, 1990.

[6] T. Jankowski, On dynamic equations with deviating arguments, *Appl. Math. Comput.,* 208:423–426, 2009.

[7] T. Jankowski, Boundary value problems for dynamic equations of Volterra type on time scales, *Dynam. Contin. Discrete Impuls. Syst. Ser. A Math. Anal.*, 16:649–659, 2009.

[8] T. Jankowski, Advanced differential equations with nonlinear boundary conditions, *J. Math. Anal. Appl.,* 304:490–503, 2005.

[9] E.R. Kaufmannm and Y.N. Raffoul, Positive solutions for a nonlinear functional dynamic equation on a time scale, *Nonlinear Anal.,* 62:1267-1276, 2005.

[10] B. Kaymakçalan, V. Lakshmikantham and S. Sivasundaram, Dynamic Systems on Measure Chains, Kluwer Academic, Boston 1996.

[11] Y. Xing, M. Han and G. Zheng, Initial value problem for first–order integro–differential equation of Volterra type on time scales, *Nonlinear Anal.,* 60:429–442, 2005.

[12] Y. Xing, M. Ding and M. Han, Periodic boundary value problems of integro–differential equation of Volterra type on time scales, *Nonlinear Anal.,* 68:127–138, 2008.