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# MONOTONE AND OSCILLATORY BEHAVIOR OF CERTAIN FOURTH ORDER NONLINEAR DYNAMIC EQUATIONS

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**ABSTRACT.** Monotone and oscillatory behavior of solutions of the fourth order dynamic equation

$$(a(x^{\Delta\Delta})^{\alpha})^{\Delta\Delta}(t)+q(t)(x^{\sigma})^{\beta}(t)=0$$

with the property that $\frac{x(t)}{\int_{t_0}^{t}\int_{t_0}^{s}a^{-1/\alpha}(\tau)\Delta\tau\Delta s}\to 0$ as $t\to\infty$ are established.

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## 1. INTRODUCTION

Consider the fourth order nonlinear dynamic equation

$$(a(x^{\Delta\Delta})^{\alpha})^{\Delta\Delta}(t)+q(t)(x^{\sigma})^{\beta}(t)=0, \tag{1.1}$$

where $\alpha$ and $\beta$ are ratios of positive odd integers, $a$ and $q$ are real-valued, positive and rd-continuous functions on a time scale $\mathbb{T}\subset\mathbb{R}$ with $\sup\mathbb{T}=\infty$, and $\int_{t_0}^{\infty}a^{-1/\alpha}(s)\Delta s=\infty$.

For completeness, we recall the following concepts related to the notion of time scales. A time scale $\mathbb{T}$ is an arbitrary nonempty closed subset of the real numbers $\mathbb{R}$, and since oscillation of solutions is our primary concern, we make the assumption that $\sup\mathbb{T}=\infty$. We assume throughout that $\mathbb{T}$ has the topology that it inherits from the standard topology on the real numbers $\mathbb{R}$. The forward and the backward jump operators are defined by

$$\sigma(t)=\inf\{s\in\mathbb{T}:s>t\}\quad\text{and}\quad\rho(t)=\sup\{s\in\mathbb{T}:s<t\},$$

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where $\inf \emptyset = \sup \mathbb{T}$ and $\sup \emptyset = \inf \mathbb{T}$ and $\emptyset$ denotes the empty set. A point $t \in \mathbb{T}$, $t > \inf \mathbb{T}$ is said to be left-dense if $\rho(t) = t$, right-dense if $t < \sup \mathbb{T}$ and $\sigma(t) = t$, left-scattered if $\rho(t) < t$ and right-scattered if $\sigma(t) > t$. A function $g : \mathbb{T} \to \mathbb{R}$ is said to be rd-continuous provided $g$ is continuous at right-dense points and at left-dense points in $\mathbb{T}$, left-hand limits exist and are finite. The graininess function $\mu$ for a time scale $\mathbb{T}$ is defined by $\mu(t) = \sigma(t) - t$, and for every function $f : \mathbb{T} \to \mathbb{R}$, the notation $f^\sigma$ denotes $f \circ \sigma$.

We recall that a solution of equation (1.1) is said to be oscillatory on $[t_0, \infty)$ in case it is neither eventually positive nor eventually negative. Otherwise, the solution is said to be nonoscillatory. Equation (1.1) is said to be oscillatory in case all its solutions are oscillatory.

Oscillatory and nonoscillatory behavior of second order nonlinear dynamic equations of the form

$$(a(x^\Delta)^\alpha)^\Delta(t) + q(t)x^\beta(t) = 0,$$

where $\alpha$, $\beta$, $a$ and $q$ are as in equation (1.1), $\alpha = 1$ or $\alpha \neq 1$, have been studied by a number of authors [7–9] and the references cited therein. To the best of our knowledge, very little is known regarding the qualitative properties of higher order dynamic equations [5, 6].

It is our aim to obtain some new criteria for the monotone and oscillatory behavior of solutions of equation (1.1) satisfying

$$\frac{x(t)}{\int_{t_0}^{t} \int_{t_0}^{s} a^{-1/\alpha}(\tau)\Delta\tau\Delta s} \to 0 \quad \text{as } t \to \infty. \tag{1.2}$$

## 2. PRELIMINARY RESULTS

For a function $f : \mathbb{T} \to \mathbb{R}$, the (delta) derivative $f^\Delta(t)$ at $t \in \mathbb{T}$ is defined to be the number (if exists) such that for all $\epsilon > 0$ there is a neighborhood $U$ of $t$ with

$$|f(\sigma(t)) - f(s) - f^\Delta(t)(\sigma(t) - s)| \leq \epsilon|\sigma(t) - s| \quad \text{for all } s \in U.$$

If the (delta) derivative $f^\Delta(t)$ exists for all $t \in \mathbb{T}$, then we say that $f$ is (delta) differentiable on $\mathbb{T}$.

We shall employ the product and quotient rules [5, Theorem 1.20] for the derivatives of the product $fg$ and the quotient $f/g$ (where $gg^\sigma \neq 0$) of two (delta) differntiable functions $f$ and $g$

$$\begin{cases} (fg)^\Delta & = f^\Delta g + f^\sigma g^\Delta = fg^\Delta + f^\Delta g^\sigma, \\ \left(\dfrac{f}{g}\right)^\Delta & = \dfrac{f^\Delta g - fg^\Delta}{gg^\sigma}, \end{cases} \tag{2.1}$$

as well as the chain rule [5, Theorem 1.90] for the derivative of the composite function $f \circ g$ for a continuously differentiable function $f : \mathbb{R} \to \mathbb{R}$ and a (delta) differentiable function $g : \mathbb{T} \to \mathbb{R}$

$$(f \circ g)^{\Delta} = \left\{ \int_0^1 f'(g + h\mu g^{\Delta}) dh \right\} g^{\Delta}. \tag{2.2}$$

For $b, c \in \mathbb{T}$ and a differentiable function $f$, the Cauchy integral of $f^{\Delta}$ is defined by

$$\int_b^c f^{\Delta}(t)\Delta t = f(c) - f(b)$$

and infinite integrals are defined as

$$\int_b^{\infty} f(t)\Delta t = \lim_{c\to\infty} \int_b^c f(t)\Delta t.$$

Note that in the case $\mathbb{T} = \mathbb{R}$, we have

$$\sigma(t) = \rho(t) = t, \quad \mu(t) = 0, \quad f^{\Delta}(t) = f'(t), \quad \int_b^c f(t)\Delta t = \int_b^c f(t)dt,$$

and in the case $\mathbb{T} = \mathbf{Z}$, we have

$$\sigma(t) = t + 1, \quad \rho(t) = t - 1, \; \mu(t) = 1, \quad f^{\Delta}(t) = \Delta f(t) = f(t + 1) - f(t)$$

and (if $b < c$)

$$\int_b^c f(t)\Delta t = \sum_{t=b}^{c-1} f(t).$$

For more discussion on time scales, we refer the reader to [5, 6, 10].

## 3. MAIN RESULTS

We shall prove the following interesting result.

**Theorem 3.1.** If $x$ is nontrivial solution of equation (1.1) such that $x(t) > 0$ for $t \geq t_0 \in \mathbb{T}$ and satisfying (1.2), then

$$x(t) > 0, \quad x^{\Delta}(t) > 0, \quad a(x^{\Delta\Delta})^{\alpha}(t) < 0, \qquad (a(x^{\Delta\Delta})^{\alpha})^{\Delta} > 0, \quad \text{for } t \geq t_0 \tag{3.1}$$

and

$$a(x^{\Delta\Delta})^{\alpha}(t), (a(x^{\Delta\Delta})^{\alpha})^{\Delta} \to 0 \quad \text{monotonically as } t \to \infty.$$

**Proof.** Let $x$ be an eventually positive solution of equation (1.1), say $x(t) > 0$ for $t \geq t_0 \in \mathbb{T}$. We claim that $(a(x^{\Delta\Delta})^{\alpha})^{\Delta}(t) > 0$ for $t \geq t_0$. To this end assume that $(a(x^{\Delta\Delta})^{\alpha})^{\Delta}(t_0) \leq 0$. Then

$$\begin{aligned}(a(x^{\Delta\Delta})^{\alpha})^{\Delta}(t) &= (a(x^{\Delta\Delta})^{\alpha})^{\Delta}(t_0) - \int_{t_0}^t q(s)(x^{\sigma})^{\beta}(s)\Delta s \\ &\leq (a(x^{\Delta\Delta})^{\alpha})^{\Delta}(t_0) := -c_1, \quad c_1 \text{ is a positive constant.}\end{aligned}$$

Integrating this inequality from $t_0$ to $t$, one can easily see that there exist a constant $c_2 > 0$ and a $t_1 \geq t_0$ such that

$$x^{\Delta\Delta}(t) \leq -c_1^{1/\alpha}(ta^{-1}(t))^{1/\alpha} \qquad \text{for } t \geq t_1.$$

Thus,

$$x(t) \leq -c\int_{t_2}^{t}\int_{t_1}^{s}(\tau a^{-1}(\tau))^{1/\alpha}\Delta\tau\Delta s,$$

where $c$ is a positive constant and for some $t_2 \geq t_1$.

Now,

$$\lim_{t\to\infty}\frac{x(t)}{\int_{t_2}^{t}\int_{t_1}^{s}(\tau a^{-1}(\tau))^{1/\alpha}\Delta\tau\Delta s} \leq -c < 0$$

which contradicts (1.2). This contradiction proves $(a(x^{\Delta\Delta})^{\alpha})^{\Delta}(t_0) > 0$. Since $t_0$ is arbitrary, we conclude that $(a(x^{\Delta\Delta})^{\alpha})^{\Delta}(t) > 0$ for $t \geq t_0$. It is now easy to see that $(a(x^{\Delta\Delta})^{\alpha})^{\Delta}(t) \to 0$ as $t \to \infty$. If this were not the case, there would exist a constant $k_1 > 0$ and $t_1 \geq t_0$. However, this implies that $x(t) > k\int_{t_2}^{t}\int_{t_1}^{s}(\tau a^{-1}(\tau))^{1/\alpha}d\tau ds$ for some constant $k > 0$ and $t_2 > t_1$, contradicting the assumption (1.2).

Next, we shall prove that $a(x^{\Delta\Delta})^{\alpha}(t) > 0$ for $t \geq t_0$. Evidently, $a(x^{\Delta\Delta})^{\alpha}(t)$ is a monotonically increasing function. If $a(x^{\Delta\Delta})^{\alpha}(t_0) > 0$, then $a(x^{\Delta\Delta})^{\alpha}(t) \geq 0$ for $t \geq t_0$ and there would exist constants $C > 0$ and $t_1 > t_0$ such that $a(x^{\Delta\Delta})^{\alpha}(t) > C_1$ for $t \geq t_1$. However, this again leads to a contradiction $x(t) > C\int_{t_2}^{t}\int_{t_1}^{s}a^{-1/\alpha}(\tau)\Delta\tau\Delta s$ for some constant $C > 0$ and some $t_2 > t_1$. Thus $a(x^{\Delta\Delta})^{\alpha}(t_0) < 0$ and $a(x^{\Delta\Delta})^{\alpha}(t) < 0$ since $t_0$ is arbitrary. Moreover, we must have $a(x^{\Delta\Delta})^{\alpha}(t) \to 0$ as $t \to \infty$, for otherwise we would again be led to a contradiction to (1.2).

Now, when $x^{\Delta\Delta}(t) < 0$ and $x(t) > 0$, we can easily see that $x^{\Delta}(t) > 0$ for $t \geq t_0$. This completes the proof. ∎

In order to characterize the behavior of solutions of equation (1.1), we may reformulate Theorem 3.1 as follows:

**Corollary 3.1.** Let x(t) be a nontrivial solution of equation (1.1) such that (1.2) hold. Then either

(i) $x$ is oscillatory on $[t_0, \infty)$, or else
(ii) $x$ satisfies the inequalities (3.1).

If $x$ is a nontrivial solution of equation (1.1) such that $x(t) \to 0$ as $t \to \infty$, it cannot satisfy the inequalities in (3.1) of Theorem 3.1. Thus, we conclude by Corollary 3.1 that $x$ is oscillatory.

Next, we let

$$Q(t) = \left(\frac{1}{a(t)}\int_{t}^{\infty}\int_{s}^{\infty}q(\tau)\Delta\tau\Delta s\right)^{1/\alpha}.$$

Now, we establish the following result when $\beta > \alpha$.

 

**Theorem 3.2.** If $\beta > \alpha$ and

$$\int_t^\infty \int_s^\infty Q(\tau)\Delta\tau\Delta s = \infty, \tag{3.2}$$

then every nontrivial solution $x$ of equation (1.1) such that (1.2) holds is oscillatory.

**Proof.** Let $x(t)$ be a nonoscillatory solution of equation (1.1) such that (1.2) holds. Assume that $x(t) > 0$ for $t \geq t_0 \in \mathbb{T}$, then (3.1) holds for $t \geq t_0$. Integrating equation (1.1) from $t$ to $u \geq t \geq t_0$ and letting $u \to \infty$ we have

$$(a(x^{\Delta\Delta})^\alpha)^\Delta(u) - (a(x^{\Delta\Delta})^\alpha)^\Delta(t) = -\int_t^u q(\tau)(x^\sigma)^\beta(\tau)\Delta\tau$$

or,

$$(a(x^{\Delta\Delta})^\alpha)^\Delta(t) \geq \int_t^\infty q(\tau)(x^\sigma)^\beta(\tau)\Delta\tau.$$

Integrating this inequality from $t$ to $u \geq t \geq t_0$ and letting $u \to \infty$ we get

$$-a(x^{\Delta\Delta})^\alpha(t) \geq \int_t^\infty \int_s^\infty q(\tau)(x^\sigma)^\beta(\tau)\Delta\tau\Delta s$$

or,

$$\begin{aligned}-x^{\Delta\Delta}(t) &\geq \left(\frac{1}{a(t)}\int_t^\infty \int_s^\infty q(\tau)\Delta\tau\Delta s\right)^{1/\alpha}(x^\sigma)^{\beta/\alpha}(t)\\ &:= Q(t)(x^\sigma)^{\beta/\alpha}(t) \quad \text{for } t \geq t_0.\end{aligned}$$

Once again, we integrate this inequality to find

$$(x^\sigma)^{-\beta/\alpha}(t)x^\Delta(t) \geq \int_t^\infty Q(\tau)\Delta\tau \quad \text{for } t \geq t_1 \geq t_0. \tag{3.3}$$

From (2.2), since $\frac{\beta}{\alpha} > 1$, we have

$$\begin{aligned}\left((x(t))^{1-\beta/\alpha}\right)^\Delta &= \left(1 - \frac{\beta}{\alpha}\right)\int_0^1 [hx^\sigma + (1-h)x]^{-\beta/\alpha}x^\Delta(t)dh\\ &\leq \left(1 - \frac{\beta}{\alpha}\right)(x^\sigma(t))^{-\beta/\alpha}x^\Delta(t),\end{aligned}$$

$$\frac{x^\Delta(t)}{(x^\sigma(t))^{\beta/\alpha}} \leq \frac{1}{(1-\frac{\beta}{\alpha})}(x^{1-\beta/\alpha}(t))^\Delta \quad \text{for } t \geq t_1. \tag{3.4}$$

Using (3.4) in (3.3) we have

$$\begin{aligned}\int_{t_1}^t \int_s^\infty Q(\tau)\Delta\tau\Delta s &\leq \frac{\alpha}{\alpha-\beta}[x^{1-\beta/\alpha}(t) - x^{1-\beta/\alpha}(t_1)]\\ &\leq \frac{\alpha}{\beta-\alpha}x^{1-\beta/\alpha}(t_1) < \infty.\end{aligned}$$

This contradicts condition (3.3) and completes the proof. ∎

The following criterion is concerned with the oscillation of all bounded solutions of equation (1.1).

**Theorem 3.3.** If condition (3.2) holds, then all bounded solutions of equation (1.1) are oscillatory.

**Proof.** Let $x(t)$ be a bounded nonoscillatory solution of equation (1.1), say $x(t) > 0$ for $t \geq t_0 \in \mathbb{T}$. There exist a constant $C > 0$ and a $t_1 \geq t_0$ such that (3.1) holds and

$$(x^\sigma)^{\beta/\alpha}(t) \geq C \quad \text{for } t \geq t_1. \tag{3.5}$$

As in the proof of Theorem 3.2, we obtain (3.3). Using (3.5) in (3.3) we have

$$x^\Delta(t) \geq C \int_t^\infty Q(\tau)\Delta\tau.$$

Integrating this inequality from $t_1$ to $t$ we get

$$x(t) \geq x(t_1) + C \int_{t_1}^t \int_s^\infty q(\tau)\Delta\tau\Delta s \to \infty \quad \text{as } t \to \infty.$$

A contradiction to the fact that $x(t)$ is bounded on $[t_0, \infty)$. This completes the proof. ∎

**Remark 3.1.** In Theorem 3.3, if $x(t)$ is not bounded and satisfies (1.2), then condition (3.2) is replaced by:

$$\limsup_{t\to\infty} \frac{1}{\int_{t_0}^t \int_{t_0}^s a^{-1/\alpha}(\tau)\Delta\tau\Delta s} \int_{t_0}^t \int_s^t Q(\tau)\Delta\tau\Delta s > 0. \tag{3.6}$$

From (3.1), there exist a constant $\theta,\ 0 < \theta < 1$ and a $t_1 \geq t_0$ so that

$$x(t) \geq \theta t x^\Delta(t) \quad \text{for } t \geq t_1. \tag{3.7}$$

Using (3.5) and (3.7) in (3.3) we see that

$$x(t) \geq \theta t x^\Delta(t) \geq \theta C t \int_t^\infty Q(\tau)\Delta\tau.$$

In the case condition (3.2) is replaced by

$$\limsup_{t\to\infty} \left( t \int_t^\infty Q(\tau)\Delta\tau \right) = \infty. \tag{3.8}$$

This condition ensures the oscillation of all bounded solutions of equation (1.1).

When $\beta < \alpha$, we obtain the following result.

**Theorem 3.4.** If $\beta < \alpha$ and

$$\int_{t_0}^\infty s^{\beta/\alpha} Q(s)\Delta s = \infty, \tag{3.9}$$

then every nontrivial solution $x$ of equation (1.1) such that (1.2) holds is oscillatory.

**Proof.** Let $x(t)$ be a nonoscillatory solution of equation (1.1) such that (1.2) holds. Assume that $x(t) > 0$ for $t \geq t_0 \in \mathbb{T}$. Then as in the proof of Theorem 3.3, we obtain (3.3) which takes the form

$$x^\Delta(t) \geq \int_t^\infty Q(s)(x^\sigma)^{\beta/\alpha}(s)\Delta s \quad \text{for } t \geq t_1 \geq t_0. \tag{3.10}$$

Set

$$u(t) = \int_t^\infty Q(s)(x^\sigma)^{\beta/\alpha}(s)\Delta s.$$

Then

$$u^\Delta(t) = -Q(t)(x^\sigma)^{\beta/\alpha}(t) \quad \text{for } t \geq t_1. \tag{3.11}$$

Using (3.7) in (3.10), we see that

$$x(t) \geq \theta t x^\Delta(t) \geq \theta t \int_t^\infty Q(s)(x^\sigma)^{\beta/\alpha}(s)\Delta s$$

$$:= \theta t u(t) \quad \text{for } t \geq t_1. \tag{3.12}$$

Using (3.12) in (3.11) we obtain

$$u^\Delta(t) \leq -Q(t)(x)^{\beta/\alpha}(t) \leq -Ct^{\beta/\alpha}Q(t)(u^\sigma)^{\beta/\alpha}(t) \quad \text{for } t \geq t_1, \tag{3.13}$$

where $C = \theta^{\beta/\alpha}$. Thus

$$(u^\sigma)^{\beta/\alpha}(t)u^\Delta(t) \leq -Ct^{\beta/\alpha}Q(t) \quad \text{for } t \geq t_1.$$

Integrating this inequality, we find

$$-\int_{t_1}^t (u^\sigma)^{\beta/\alpha}(s)u^\Delta(s)\Delta s \geq C\int_{t_1}^t s^{\beta/\alpha}Q(s)\Delta s. \tag{3.14}$$

From (2.2) and $\frac{\beta}{\alpha} < 1$, we have

$$(u^{1-\beta/\alpha}(t))^\Delta = \left(1 - \frac{\beta}{\alpha}\right)\int_0^1 [hu^\sigma + (1-h)u]^{-\beta/\alpha}u^\Delta dh$$

$$\geq \left(1 - \frac{\beta}{\alpha}\right)(u^\sigma)^{-\beta/\alpha}(t)u^\Delta(t) \quad \text{for } t \geq t_1. \tag{3.15}$$

Using (3.15) in (3.14) we obtain a contradiction to (3.9). This completes the proof. ∎

For illustration we consider the following example

**Example 3.1.** Here, we shall reformulate results which are sufficient conditions for the oscillation of equation (1.1).

If $\mathbb{T} = \mathbb{R}$, then conditions (3.2) and (3.8), respectively become,

$$\int^\infty \int_s^\infty Q(\tau)d\tau ds = \infty \tag{3.3$'$}$$

and

$$\int^\infty s^{\beta/\alpha}Q(s)ds = \infty, \tag{3.9$'$}$$

where

$$Q(t) = \left(\frac{1}{a(t)}\int_r^\infty \int_s^\infty q(\tau)d\tau ds\right)^{1/\alpha}.$$

We note that conditions $(3.3)'$ and $(3.9)'$ are new.

If $\mathbb{T} = \mathbf{Z}$, then conditions (3.2) and (3.8), respectively, become

$$(3.3)'' \qquad \sum_{n=n_0}^{\infty} \sum_{j=n+1}^{\infty} Q(j) = \infty$$

and

$$(3.9)'' \qquad \sum_{j=n_0}^{\infty} j^{\beta/\alpha} Q(j) = \infty$$

where

$$Q(n) = \left( \frac{1}{a(n)} \sum_{j=n+1}^{\infty} \sum_{s=j+1}^{\infty} q(s) \right)^{1/\alpha}.$$

We note that condition $(3.3)''$ and $(3.9)''$ are new.

We may employ other types of time scales e.g. $\mathbb{T} = h\mathbf{Z}$ with $h > 0$, $\mathbb{T} = q^{\mathbb{N}_0}$ with $q > 1$, $\mathbb{T} = \mathbb{N}_0^2, \ldots$ etc., see [5, 6]. The details are left to the reader.

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