ISSN: 1748-0345 (Online) www.tagajournal.com

EIGENVALUE PROBLEM FOR ODES WITH A PERTURBED Q-LAPLACE OPERATOR

DONAL O'REGAN AND ALEKSANDRA ORPEL

Department of Mathematics, National University of Ireland, Galway, Ireland
Faculty of Mathematics and Computer Science, University of Lodz, Poland

ABSTRACT. We investigate the eigenvalue interval for boundary value problem with a one-dimensional perturbed q-Laplace operator. Our results cover also the case when the right-hand side has singularities. Applying variational methods we prove the existence of positive solutions and establish their continuous dependence on functional parameters.

AMS (MOS) Subject Classification. 34B16.

1. Introduction

This paper is devoted to the eigenvalue problem associated with a second order ODE containing a perturbed one-dimensional q-Laplace operator with a singularity at 0. Our main goal is to discuss when the equation

$$-\left(\left(a(t)\left|u'(t)\right|^{q-2}u'\right)' + \frac{ka(t)}{t}\left|u'(t)\right|^{q-2}u'\right) = f_1(t,u(t)) + \lambda f_2(t,u(t)) \tag{1.1}$$

a.e. in $(0,T)$, where $q \geq 2$, $k > 1$, $T > 0$, $a \in C^1([0,T])$, possesses at least one positive solution satisfying the boundary conditions

$$u'(0) = 0 \text{ and } u(T) = 0. \tag{1.2}$$

Our paper is motivated by the large number of papers associated with similar problems, see for example [1], [2], [3], [4], [5], [6], [8], [9]. The majority of these papers discuss the case $q = 2$ or when the right-hand side of (1) has a special form. The approach presented here is based on methods in calculus of variations. Thus we treat (1.1)–(1.2) as the Euler-Lagrange equation for the following functional

$$J(u) = \int_0^T t^k \left(-F_\lambda(t,u(t)) + \frac{1}{q}a(t)\left|u'(t)\right|^q\right) dt, \tag{1.3}$$

where $F_\lambda(t,u) := \int_0^u (\overline{f_1}(t,l) + \lambda\overline{f_2}(t,l))dl$ and for $i = 1, 2$,

$$\overline{f}_i(t,u) = \begin{cases} f_i(t,u) & \text{if } u \in [0,d_1], t \in [0,T] \\ +\infty & \text{if } u \in R\backslash[0,d_1], t \in [0,T] \end{cases}$$

with positive $d_1 \in I := (-b, c)$, where b and c are fixed positive numbers. We deal with the case when the following assumptions hold

f1 $f_1, f_2 : [0,T] \times I \to R$ *are Caratheodory functions,* λ *is real number* R *such that for almost all* $t \in [0,T]$ *and all* $u \in I$

$$f_1(t,u) + \lambda f_2(t,u) \geq 0$$

and $t \mapsto f_1(t,0) + \lambda f_2(t,0)$ *is not identically zero in a certain subset of* $[0,T]$ *with positive measure.*

f2 *there exists positive* $d \in I$ *such that for* $i = 1, 2$, $u \mapsto f_1(t,u) + \lambda f_2(t,u)$ *is increasing in* I *for a.a.* $t \in [0,T]$, *and*

$$\max_{u \in [0,d]} (f_1(\cdot, u) + \lambda f_2(\cdot, u)) \in L^{q'}(0,T),$$

with $q' = \frac{q}{q-1}$.

f3 $a \in C^1([0,T])$ and $a_{\min} := \min_{t \in [0,T]} a(t) > 0$.

Let

$$\begin{aligned} \widetilde{U} \;=\; &\big\{ u \in C^1([0,T]) : u(T) = 0 \text{ and } u'(0) = 0 \text{ and } u'(t) < 0 \\ &\text{for } t \in [0,T] \text{ and } t^k a(t) \left|u'\right|^{q-2} u' \in A([0,T]) \Big\}, \end{aligned} \tag{1.4}$$

where $A([0,T])$ denotes the space of absolutely continuous functions v such that $v'/t^k \in L^{q'}(0,T)$.

Let us note that in this case J is not necessarily either bounded or continuously differentiable in its natural domain. Therefore we describe the set denoted by U in which J is bounded below and possesses a positive minimizer $\overline{u} \in U$. The special properties of U and the Fenchel equalities for auxiliary functionals allow us to show that $\overline{u}$ is the solution of (1.1)–(1.2). Also in this paper we discuss the continuous dependence of solutions on functional parameters for our problem. Here we employ the schema presented e.g. in [7], [6]. Roughly we prove that a sequence of solutions $(u_m)_{m \in N}$ of the problem

$$\left\{ \begin{aligned} &-\left(\left(a(t) \left|u'(t)\right|^{q-2} u' \right)' + \tfrac{k a(t)}{t} \left|u'(t)\right|^{q-2} u' \right) \\ &= f_1(t, u(t), w(t)) + \lambda f_2(t, u(t), z(t)) \; a.e. \text{ in } (0,T), \\ &u'(0) = 0 \text{ and } u(T) = 0, \end{aligned} \right. \tag{1.5}$$

corresponding to the sequence of parameters $((w_m, z_m))_{m \in N} \subset L^{p_1}(0,T) \times L^{p_2}(0,T)$, where $p_1, p_2 > 2$, tends uniformly to $\overline{u}$ in $[0,T]$ (up to a subsequence) provided that the sequence of parameters tends almost everywhere in $(0,T)$ to $(w_0, z_0) \in L^{p_1}(0,T) \times L^{p_2}(0,T)$. Moreover we show that $\overline{u}$ is the solution of (1.5) with parameters (w_0, z_0).

Lemma 1.1. *Assume f1, f2 and f3. If u is a solutions of (1.1)–(1.2) such that $u(t) \in I$ then $u'(t) < 0$ for $t \in (0,T)$.*

Proof. Let $h(t) := t^k a(t)|u'(t)|^{q-2}u'(t)$ for all $t \in [0,T]$. Since $h'(t) < 0$ for all $t \in (0,T)$ we see that h is decreasing. Moreover $h(0) = 0$, so we have $h(t) < h(0) = 0$ for $t \in (0,T)$. Therefore, by f3 and definition of h, we see that $u'(t) < 0$ for $t \in (0,T)$. □

Lemma 1.2. *Suppose that f1, f2, f3 hold and assume additionally that for $d \in I$ defined in f2 the following inequality hold*

f4
$$\int_0^T \left(\frac{1}{a(s)s^k}\int_0^s r^k\left(f_1(r,d)+\lambda f_2(r,d)dr\right)\right)^{\frac{1}{q-1}} ds \le d.$$

Then the set $U := \{u \in \widetilde{U}; u(t) \le d \text{ for all } t \in [0,T]\}$ has the following property: for each $u \in U$ there exists $\widetilde{u} \in U$ such that for a.e. in $(0,T)$

$$-\left(a(t)t^k\left|\widetilde{u}'(t)\right|^{q-2}\widetilde{u}'(t)\right)' = t^k(f_1(t,u(t))+\lambda f_2(t,u(t))). \tag{1.6}$$

Proof. Fix $u \in U$. We show that

$$\widetilde{u}(t) = \int_t^T \left(\frac{1}{a(s)s^k}\int_0^s r^k\left(f_1(r,u(r))+\lambda f_2(r,u(r))dr\right)\right)^{\frac{1}{q-1}} ds$$

also belongs to U and satisfies (1.6). To this end we note

$$\widetilde{u}'(t) = -\left(\frac{1}{a(t)t^k}\int_0^t r^k\left(f_1(r,u(r))+\lambda f_2(r,u(r))dr\right)\right)^{\frac{1}{q-1}}$$

and further

$$\begin{aligned} &a(t)t^k\left|\widetilde{u}'(t)\right|^{q-2}\widetilde{u}'(t) \\ &= -a(t)t^k\left[\left(\frac{1}{a(t)t^k}\int_0^t r^k\left(f_1(r,u(r))+\lambda f_2(r,u(r))dr\right)\right)^{\frac{1}{q-1}}\right]^{q-1} \\ &= -\int_0^t r^k\left(f_1(r,u(r))+\lambda f_2(r,u(r))dr\right) \end{aligned}$$

which gives (1.6). It is clear that $\widetilde{u}(T) = 0$, $\widetilde{u} \in C([0,T]) \cap C^1((0,T])$. Moreover, by Hőlder's inequality, we have

$$\begin{aligned} |\widetilde{u}'(t)|^{q-1} &= \frac{1}{a(t)t^k}\int_0^t r^k f_1(r,u(r))+\lambda f_2(r,u(r))dr \\ &\le \frac{1}{a(t)t^k}\left(\int_0^t l^{qk}dl\right)^{1/q}\left(\int_0^t (\overline{f_1}(l,u(l))+\lambda\overline{f_2}(t,u(t)))^{q'}dl\right)^{1/q'} \\ &\le \frac{1}{a(t)t^k}\left(\frac{1}{qk+1}\right)^{1/q} t^{k+1/q}\left(\int_0^T (\overline{f_1}(l,d)+\lambda\overline{f_2}(t,d))^{q'}dl\right)^{1/q'} \end{aligned}$$

$$\leq \frac{1}{a_{\min}}\left(\frac{1}{qk+1}\right)^{1/q}\left(\int_0^T (\overline{f_1}(l,d)+\lambda\overline{f_2}(t,d))^{q'}dl\right)^{1/q'} t^{1/q}.$$

Therefore

$$\lim_{t\to 0^+}\widetilde{u}'(t)=0.$$

Taking into account (1.6) we get

$$\left(a(t)t^k\left|\widetilde{u}'(t)\right|^{q-2}\widetilde{u}'(t)\right)'/t^k=(f_1(t,u(t))+\lambda f_2(t,u(t)))$$

which means, by f2, that $\left(a(t)t^k\left|\widetilde{u}'(t)\right|^{q-2}\widetilde{u}'(t)\right)'/t^k$ belongs to $\in L^{q'}(0,T)$. Finally, by the definition of $\widetilde{u}$ we get $a(t)t^k\left|\widetilde{u}'(t)\right|^{q-2}\widetilde{u}'(t)\in A([0,T])$. □

Theorem 1.3. *Assume that (f1)–(f4) hold. If $(u_m)_{m\in N}\subset U$ is a minimizing sequence of the functional $J:U\to R$ then there exists a sequence $(v_m)_{m\in N}\subset W^{1,q'}(0,T)$ such that*

$$-v_m'(t)=t^k\left(\overline{f_1}(t,u_m))+\lambda\overline{f_2}(t,u_m)\right)\ \ a.e.\ \ in\ (0,1) \tag{1.7}$$

and

$$\lim_{m\to\infty}\int_0^T\frac{1}{q'(t^ka(t))^{\frac{q'}{q}}}|v_m(t)|^{q'}+\frac{1}{q}a(t)t^k|u_m'(t)|^q-u_m'(t)v_m(t))dt=0. \tag{1.8}$$

Proof. Let us note that J is bounded below on U. Indeed, for each $u\in U$ one can see

$$\begin{aligned} J(u) &= \int_0^T\left[-t^kF_\lambda(t,u)+\frac{a(t)t^k}{q}\left|u'(t)\right|^q\right]dt \\ &\geq -\int_0^T t^kF_\lambda(t,u(t))dt\geq -\int_0^T t^ku(t)[\left(\overline{f_1}(t,d))+\lambda\overline{f_2}(t,d)\right)]dt \\ &\geq -dT^k\int_0^T[\left(\overline{f_1}(t,d))+\lambda\overline{f_2}(t,d)\right)]dt, \end{aligned} \tag{1.9}$$

and further $-\infty<\min:=\inf_{u\in\widetilde{U}}J(u)<+\infty$, which implies that for each $\varepsilon>0$ there exists $m_0\in N$ such that $J(u_m)<\varepsilon+\min$ for all $m\geq m_0$. Taking into account Lemma 1.2 we infer that for each $u_m\in U$, there exists $(\overline{u}_m)_{m\in N}\subset U$ such that

$$\begin{gathered} -(t^ka(t)|\overline{u}_m'(t)|^{q-2}\overline{u}_m'(t))'=t^k\left(\overline{f_1}(t,u_m(t))+\lambda\overline{f_2}(t,u_m(t))\right)\ \text{a.e. in }(0,T) \\ \overline{u}_m'(0)=0\text{ and }\overline{u}_m(T)=0. \end{gathered} \tag{1.10}$$

We consider the following sequence $(v_m)_{m\in N}\subset W^{1,q'}(0,T)$

$$v_m(t):=t^ka(t)\left|\overline{u}_m'(t)\right|^{q-2}\overline{u}_m'(t)\text{ for }t\in(0,T) \tag{1.11}$$

and note, by (1.10), that

$$\begin{aligned} -v_m'(t) &\in \partial_u\{t^kF_\lambda(t,u_m(t))\} \\ &= \{t^k\left(\overline{f_1}(t,u_m(t))+\lambda\overline{f_2}(t,u_m(t))\right)\}\ \text{a. e. in }(0,T) \end{aligned} \tag{1.12}$$

which can be rewritten as (1.7).

Moreover, by the Fenchel equality for $L^q(0,T) \ni u \mapsto \int_0^T t^k F_\lambda(t,u(t))dt$, we infer that for each $m \geq m_0$

$$\begin{aligned} \min + \varepsilon &> J(u_m) \\ &= \int_0^T t^k F_\lambda^*\left(t, -\frac{v_m'(t)}{t^k}\right)dt + \int_0^T u_m(t)v_m'(t)dt \\ &\quad + \int_0^T \frac{a(t)t^k}{q}|u_m'(t)|^q dt, \end{aligned} \tag{1.13}$$

where $F_\lambda^*(t,v) := \sup_{u\in R}(uv - F_\lambda(t,u))$ for all $(t,v,\lambda) \in (0,T)\times R \times R$.

On the other hand, for all $u \in U$, we have the estimate

$$\begin{aligned} \min = \inf_{u\in\widetilde{U}} J(u) &\leq \int_0^T \frac{a(t)t^k}{q}|u'(t)|^q dt - \int_0^T t^k F_\lambda(t,u(t))dt \\ &\leq \int_0^T \frac{a(t)t^k}{q}|u'(t)|^q dt + \int_0^T t^k F_\lambda^*\left(t, -\frac{v_m'(t)}{t^k}\right)dt - \int_0^T u'(t)v_m(t)dt. \end{aligned}$$

Therefore one sees

$$\begin{aligned} \min &\leq \inf_{u\in\widetilde{U}}\left[\int_0^T \frac{a(t)t^k}{q}|u'(t)|^q dt + \int_0^T t^k F_\lambda^*\left(t, -\frac{v_m'(t)}{t^k}\right)dt \right. \\ &\quad \left. - \int_0^T u'(t)v_m(t)dt\right] = \int_0^T t^k F_\lambda^*\left(t, -\frac{v_m'(t)}{t^k}\right)dt \\ &\quad - \sup_{u\in\widetilde{U}}\left[\int_0^T u'(t)v_m(t)dt - \int_0^T \frac{a(t)t^k}{q}|u'(t)|^q dt\right] \end{aligned} \tag{1.14}$$

for all $m \in N$. Now, by formula (1.11) and the properties of U we have

$$\begin{aligned} \int_0^T \frac{1}{q'(t^k a(t))^{\frac{q'}{q}}}|v_m(t)|^{q'}dt &= \int_0^T \overline{u}_m'(t)v_m(t)dt - \int_0^T \frac{a(t)t^k}{q}|u'(t)|^q dt \\ &\leq \sup_{u\in\widetilde{U}}\left[\int_0^T u'(t)v_m(t)dt - \int_0^T \frac{a(t)t^k}{q}|u'(t)|^q dt\right] \\ &\leq \sup_{z\in L^2(0,T)}\left[\int_0^T z(t)v_m(t)dt - \int_0^T \frac{a(t)t^k}{q}|z(t)|^q dt\right] \\ &= \int_0^T \frac{1}{q'(t^k a(t))^{\frac{q'}{q}}}|v_m(t)|^{q'}dt, \end{aligned}$$

which implies

$$\sup_{u\in\widetilde{U}}\left[\int_0^T u'(t)v_m(t)dt - \int_0^T \frac{t^k}{2}|u'(t)|dt\right] = \int_0^T \frac{1}{q'(t^k a(t))^{\frac{q'}{q}}}|v_m(t)|^{q'}dt, \tag{1.15}$$

that for all $m \in N$. Consequently, (1.14) yields that

$$\min \leq \int_0^T t^k F_\lambda^*\left(t, -\frac{v_m'(t)}{t^k}\right)dt - \int_0^T \frac{1}{q'(t^k a(t))^{\frac{q'}{q}}}|v_m(t)|^{q'}dt, \text{ for all } m \in N. \tag{1.16}$$

Combining (1.13) and (1.16) we obtain the estimate

$$
\begin{aligned}
0 &\le \left(\int_0^T \frac{a(t)t^k}{q}|u_m'(t)|^q dt + \int_0^T \frac{1}{q'(t^k a(t))^{\frac{q'}{q}}}|v_m(t)|^{q'} dt - \int_0^T u_m'(t)v_m(t)dt \right) \\
&= \left\{ \int_0^T \frac{1}{q'(t^k a(t))^{\frac{q'}{q}}}|v_m(t)|^{q'} dt - \int_0^T t^k F_\lambda^*(t, -\frac{v_m'(t)}{t^k})dt \right\} \\
&+ \left\{ \int_0^T \frac{a(t)t^k}{q}|u'(t)|^q dt + \int_0^T u_m(t)v_m'(t)dt + \int_0^T t^k F_\lambda^*(t, -\frac{v_m'(t)}{t^k})dt \right\} \\
&\le -\min + \min + \varepsilon = \varepsilon,
\end{aligned}
$$

for all $m \ge m_0$. Since $\varepsilon > 0$ was arbitrary, we get (1.8). □

Theorem 1.4. *If (f1)–(f4) hold, then problem* (1.1)–(1.2) *possesses at least one solution $\overline{u} \in U$ which is a minimizer of $J : U \to R$.*

Proof. We start our proof with the observation that for $a \in R$ large enough the set $S_a := \{u \in U, J(u) \le a\}$ is nonempty. Let $(u_m)_{m\in N} \subset S_a$ be a minimizing sequence of $J : U \to R$. Taking into account the estimate (1.9), we see that $(t^{k/q}u_m')_{m\in N}$ is bounded in the $L^q(0,T)$-norm, and further $(\left(t^k u_m\right)')_{m\in N}$ is bounded in the $L^q(0,T)$-norm. Thus, going if necessary to a subsequence, $(t^k u_m)_{m\in N}$ is weakly convergent in $W_0^{1,q}(0,T)$ to a certain $\widetilde{z} \in W_0^{1,q}(0,T)$ and, as a consequence, it is uniformly convergent in $[0,T]$. Moreover $(u_m)_{m\in N}$ is bounded in $L^q(0,T)$ so up to a subsequence, $(u_m)_{m\in N}$ tends weakly to a certain $\overline{u} \in L^q(0,T)$. Therefore $\widetilde{z}(t) = t^k\overline{u}(t)$ and further $\overline{u}$ is continuous in $(0,T]$ and $0 \le \overline{u} \le d$ in $(0,T]$. Now we show that $\overline{u}' < 0$ and $\overline{u} \in C^1([0,T])$. To this end we see, by Theorem 1.3, that there exists a sequence $(v_m)_{m\in N} \subset W^{1,q'}(0,T)$ such that

$$
-v_m'(t) = t^k \left(f_1(t, u_m(t)) + \lambda \overline{f_2}(t, u_m(t))\right), \text{ for a.e. } t \in (0,T), \tag{1.17}
$$

and such that

$$
\lim_{m\to\infty} \int_0^T \left(\frac{1}{q'(t^k a(t))^{\frac{q'}{q}}} |v_m(t)|^{q'} + \frac{a(t)t^k}{q} |u_m'(t)|^q - u_m'(t)v_m(t) \right) dt = 0. \tag{1.18}
$$

Assertion (1.17) leads to the conclusion that $(v_m'/t^k)_{m\in N}$ and $(v_m')_{m\in N}$ are bounded in the $L^{q'}(0,T)$ norm, which implies the weak convergence (up to subsequences) of $(v_m')_{m\in N}$ and $(v_m'/t^k)_{m\in N}$ in $L^{q'}(0,T)$. By (1.18) we can deduce also the boundedness of $(v_m)_{m\in N}$ in $L^{q'}(0,T)$. Finally, going if necessary to a subsequence, $(v_m)_{m\in N}$ is weakly convergent in $W^{1,q'}(0,T)$ to $\overline{v} \in W^{1,q'}(0,T)$. Therefore $(v_m)_{m\in N}$ tends uniformly to $\overline{v}$ in $[0,T]$. Since for all $m \in N$, v_m is continuous and nonpositive, we obtain the continuity and positivity of $\overline{v}$. Our task is now to prove that

$$
\overline{v}'(t) = -t^k(f_1(t, \overline{u}(t)) + \lambda \overline{f_2}(t, \overline{u}(t))) \text{ a.e. in } (0,T) \tag{1.19}
$$

$$
\overline{v}(t) = t^k a(t) \left|\overline{u}'(t)\right|^{q-2} \overline{u}'(t) \text{ a.e. in } (0,T). \tag{1.20}
$$

To this end one notes, by (1.17) and the properties of $(u_m)_{m\in N}$ and $(v'_m)_{m\in N}$,

$$
\begin{aligned}
0 &\geq \liminf_{m\to\infty}\int_0^T \left(v'_m(t)u_m(t) + t^k F_\lambda^*\left(t, -\frac{v'_m(t)}{t^k}\right) + t^k F_\lambda(t, u_m(t))\right) dt \\
&\geq \int_0^T \left(\overline{v}'(t)\overline{u}(t) + t^k F_\lambda^*\left(t, -\frac{\overline{v}'(t)}{t^k}\right) + t^k F_\lambda(t, \overline{u}(t))\right) dt \geq 0,
\end{aligned}
$$

where the last inequality is due to the properties of the Fenchel conjugate. Thus we get

$$
\overline{v}'(t) = -t^k\left(f_1(t,\overline{u}(t)) + \lambda f_2(t,\overline{u}(t))\right) \text{ a.e. in } (0,T). \tag{1.21}
$$

On the other hand, (1.18) gives

$$
\begin{aligned}
0 &\geq \liminf_{m\to\infty}\int_0^T \left(\frac{1}{q'(t^k a(t))^{\frac{q'}{q}}}\left|v_m(t)\right|^{q'} + \frac{a(t)t^k}{q}\left|u'_m(t)\right|^q - u'_m(t)v_m(t)\right) dt \\
&\geq \int_0^T \left(\frac{1}{q'(t^k a(t))^{\frac{q'}{q}}}\left|\overline{v}(t)\right|^{q'} + \frac{a(t)t^k}{q}\left|\overline{u}'(t)\right|^q - \overline{u}'(t)\overline{v}(t)\right) dt \geq 0.
\end{aligned}
$$

Consequently, applying again the properties of the Fenchel transform, we get

$$
\overline{v}(t) = t^k a(t)\left|\overline{u}'(t)\right|^{q-2}\overline{u}'(t) \text{ a.e. in } (0,T). \tag{1.22}
$$

Summarizing, assertions (1.21) and (1.22) give

$$
(t^k a(t)\left|\overline{u}'(t)\right|^{q-2}\overline{u}'(t))' = -t^k f_1(t,\overline{u}(t)) + \lambda f_2(t,\overline{u}(t)) \text{ for a. a. } t\in(0,T) \tag{1.23}
$$

which can be rewritten as (1.1)–(1.2). Moreover it is clear that Lemmas 1.1 and 1.2 yield $\overline{u}\in C^1([0,T])$, $\overline{u}'(0)=0$, $\overline{u}(T)=0$, $\overline{u}'<0$ a.e. in $[0,T]$, $t^k a(t)\left|\overline{u}'(t)\right|^{q-2}\overline{u}'(t)\in L^{q'}(0,T)$. Finally $\overline{u}\in U$.

Finally, by the uniform convergence of $(u_m)_{m\in N}$ to $\overline{u}$ and the weak convergence of $(t^{k/q}u'_m)_{m\in N}$ in $L^q(0,T)$ to $t^{k/q}\overline{u}'$, one gets

$$
\begin{aligned}
\inf_{u\in U} J(u) &= \liminf_{m\to\infty}\int_0^T t^k\left(-F_\lambda(t,u_m(t)) + \frac{a(t)}{q}\left|u'_m(t)\right|^q\right) dt \\
&\geq \int_0^T t^k\left(-F_\lambda(t,\overline{u}(t)) + \frac{a(t)}{q}\left|\overline{u}'(t)\right|^q\right) dt = J(\overline{u}).
\end{aligned}
$$

□

2. Stability of solutions

In this section we shall investigate the dependence on functional parameters. Let us consider the set $W\times Z\subset L^{p_1}(0,T)\times L^{p_2}(0,T)$, where $p_1,p_2>2$. We start with assumptions which guarantee that for each pair $(w,z)\in W\times Z$ there exists at least one positive and decreasing solution of (1.5). For this we assume f1p *$f_1 : [0,T] \times I \times R \to R$, $f_2 : [0,T] \times I \times R \to R$ are Caratheodory functions, λ is real number R such that for almost all $t \in [0,T]$ and all $u \in I$, $(x,y) \in R^2$*

$$f_1(t,u,x) + \lambda f_2(t,y) \geq 0;$$

f2p *there exists positive $d \in I$ such that for each $(w,z) \in W \times Z$, $u \mapsto f_1(t,u,w(t)) + \lambda f_2(t,u,z(t))$ is increasing in I for a.a. $t \in [0,T]$, and*

$$\max_{u\in[0,d]} \left(f_1(\cdot,u,w(\cdot)) + \lambda f_2(\cdot,u,z(\cdot))\right) \in L^{q'}(0,T),$$

with $q' = \frac{q}{q-1}$, and *$t \mapsto f_1(t,0,w(t)) + \lambda f_2(t,0,z(t))$ is not identically zero in a certain subset of $[0,T]$ with positive measure.*

f4p *for each $(w,z) \in W \times Z$*

$$\int_0^T \left(\frac{1}{a(s)s^k}\int_0^s r^k \left(f_1(r,d,w(r)) + \lambda f_2(r,d,z(r))dr\right)\right)^{\frac{1}{q-1}} ds \leq d.$$

f5p *there exists $M > 0$ such that for each $(w,z) \in W \times Z$*

$$\int_0^T t^k \max_{u\in[0,d]} [f_1(t,u,w(t)) + \lambda f_2(t,u,z(t))]dt \leq M.$$

Theorem 2.1. *Suppose that (f1p), (f2p), (f4p), (f5p) and (f3) hold. Consider the sequence of parameters $(w_m,z_m)_{m\in N} \in W \times Z$such that for each $m \in N$, we denote by $u_m \in U$ a solution of (1.5). If $(w_m,z_m)_{m\in N}$ tends a.e. in $[0,T]$ to (w_0,z_0), then the sequence of solutions $(u_m)_{m\in N}$ tends uniformly (up to a subsequence) to a certain $u_0 \in U$ being a solution of (1.5) for parameters (w_0,z_0).*

Proof. By the previous theorem for each pair $(w_m,z_m)_{m\in N} \in W \times Z$ there exists a solution $u_m \in U$ for problem (1.5), namely

$$-\left(a(t)t^k \left|u_m'(t)\right|^{q-2} u_m'(t)\right)' = t^k f_1(t,u_m(t),w_m(t)) + \lambda f_2(t,u_m(t),z_m(t)). \tag{2.1}$$

Thus we have

$$\begin{aligned}
&\int_0^T t^k \left|u_m'(t)\right|^q dt \\
&\leq \frac{1}{a_{\min}}\int_0^T a(t)t^k \left|u_m'(t)\right|^{q-2} u_m'(t)u_m'(t)dt \\
&= \frac{1}{a_{\min}}\int_0^T a(t)t^k \left|u_m'(t)\right|^{q-2} u_m'(t)u_m'(t)dt \\
&= \frac{1}{a_{\min}}\left(\left[u(t)a(t)t^k \left|u_m'(t)\right|^{q-2} u_m'(t)\right]_0^T \right.\\
&\qquad \left. - \int_0^T \left(a(t)t^k \left|u_m'(t)\right|^{q-2} u_m'(t)\right)' u_m(t)dt\right) \\
&= \frac{1}{a_{\min}}\int_0^T -\left(a(t)t^k \left|u_m'(t)\right|^{q-2} u_m'(t)\right)' u_m(t)dt
\end{aligned}$$

$$= \frac{1}{a_{\min}} \int_0^T t^k \max_{u\in[0,d]} [f_1(t,u,w_m(t)) + \lambda f_2(t,u,z_m(t))]dt \leq \frac{M}{a_{\min}}.$$

Therefore we see that $(t^{k/q}u'_m)_{m\in N}$ is bounded in the $L^q(0,T)$-norm, and further $((t^k u_m)')_{m\in N}$ is bounded in the $L^q(0,T)$-norm. Now, employing a reasoning similar to that in the proof of Theorem 1.4, we infer that $(t^k u_m)_{m\in N}$ tends weakly (up to a subsequence) in $W_0^{1,q}(0,T)$ to a certain $x_0 \in W_0^{1,q}(0,T)$. Consequently, it is uniformly convergent in $[0,T]$. On the other hand $(u_m)_{m\in N}$ is bounded in $L^q(0,T)$ so up to a subsequence, $(u_m)_{m\in N}$ is weakly convergent to a certain $u_0 \in L^q(0,T)$. Therefore $x_0(t) = t^k u_0(t)$ and further u_0 is continuous in $(0,T]$ and $0 \leq u_0 \leq d$ in $(0,T]$. Now we prove that $u'_0 < 0$ and $u_0 \in C^1([0,T])$. For this we consider the sequence

$$v_m(t) = t^k a(t) \left|u'_m(t)\right|^{q-2} u'_m(t) \text{ a.e. in } (0,T).$$

By (2.1),

$$-v'_m(t) = t^k f_1(t,u_m(t),w_m(t)) + \lambda f_2(t,u_m(t),z_m(t)) \text{ a.e. in } (0,T). \tag{2.2}$$

The above assertions and the properties of the sequence $(u_m)_{m\in N}$ guarantee that $(v_m)_{m\in N}$ is bounded in $W^{q'}(0,T)$ and further, it is weakly convergent (up to a subsequence) to $v_0 \in W^{q'}(0,T)$. Finally $(v_m)_{m\in N}$ is uniformly convergent to v_0 in $[0,T]$. Since each $v_m(t) < 0$ we see that $v_0(t) \leq 0$ in $(0,T)$ and $v_0 \in C([0,T])$. Moreover we have

$$\begin{aligned} 0 &= \liminf_{m\to\infty} \int_0^T \left(\frac{1}{q'(t^k a(t))^{\frac{q'}{q}}} \left|v_m(t)\right|^{q'} \right. \\ &\quad \left. + \frac{a(t)t^k}{q} \left|u'_m(t)\right|^q - u'_m(t)v_m(t) \right) dt \\ &\geq \int_0^T \left(\frac{1}{q'(t^k a(t))^{\frac{q'}{q}}} \left|v_0(t)\right|^{q'} + \frac{a(t)t^k}{q} \left|u'_0(t)\right|^q - u'_0(t)v_0(t) \right) dt \geq 0. \end{aligned} \tag{2.3}$$

We now show

$$\begin{aligned} 0 &= \liminf_{m\to\infty} \int_0^T \left(v'_m(t)u_m(t) + t^k F^*_\lambda \left(t, -\frac{v'_m(t)}{t^k}, w_m(t), z_m(t)\right) \right. \\ &\quad \left. + t^k F_\lambda(t,u_m(t),w_m(t),z_m(t)) \right) dt \\ &\geq \int_0^T \left(v'_0(t)u_0(t) + t^k F^*_\lambda \left(t, -\frac{v'_0(t)}{t^k}, w_0(t), z_0(t)\right) \right. \\ &\quad \left. + t^k F_\lambda(t,u_0(t),w_0(t),z_0(t)) \right) dt \geq 0, \end{aligned} \tag{2.4}$$

where for almost all $t \in [0,T]$ and all $u \in I$, $(x,y) \in R^2$ and $v^* \in R$,

$$F_\lambda(t,u,x,y) := \int_0^u (\overline{f_1}(t,l,x) + \lambda \overline{f_2}(t,l,y))dl,$$

$$F_\lambda^*(t, v^*, x, y) := \sup_{u \in R} (uv^* - F_\lambda(t, u, x, y))$$

with

$$\overline{f}_1(t,u,x) = \begin{cases} f_1(t,u,x) \text{ if } u \in [0,d_1],\ t \in [0,T] \\ +\infty \text{ if } u \in R \backslash [0,d_1],\ t \in [0,T] \end{cases}$$

$$\overline{f}_2(t,u,y) = \begin{cases} f_2(t,u,y) \text{ if } u \in [0,d_1],\ t \in [0,T] \\ +\infty \text{ if } u \in R \backslash [0,d_1],\ t \in [0,T]. \end{cases}$$

For this we note that (2.2), convexity of F_λ with respect to the second variable and definition of F_λ yield

$$-\frac{v_m'(t)}{t^k} \in \partial_u F_\lambda(t, u_m(t), w_m(t), z_m(t)))$$

for a.a. $t \in (0,T)$ and all $m \in N$, where $\partial_u F_\lambda$ is the subdifferential of F_λ with respect to the second variable:

$$\partial_u F_\lambda(t,u,x,y) := \{v^* \in R, F_\lambda(t,v,x,y) \geq F_\lambda(t,u,x,y) + v^*(v-u) \quad \text{for all } v \in R\}.$$

Now applying the Fenchel equality for the function $F_\lambda(t, \cdot, x, y)$ we get

$$(v_m'(t)u_m(t) + t^k F_\lambda^* \left(t, -\frac{v_m'(t)}{t^k}, w_m(t), z_m(t)\right) + t^k F_\lambda(t, u_m(t), w_m(t), z_m(t))) = 0$$

for a.a. $t \in (0,T)$ and all $m \in N$. Thus

$$\lim_{m\to\infty} \int_0^T \left(v_m'(t)u_m(t) + t^k F_\lambda^* \left(t, -\frac{v_m'(t)}{t^k}, w_m(t), z_m(t)\right) \right.$$
$$\left. + t^k F_\lambda(t, u_m(t), w_m(t), z_m(t)) \right) dt = 0.$$

On the other hand, by the assumptions on F_λ and properties of the sequences, we know that

(2.5) $$\lim_{m\to\infty} \int_0^T v_m'(t)u_m(t)dt = \int_0^T v_0'(t)u_0(t)dt$$

and

(2.6) $$\lim_{m\to\infty} \int_0^T t^k F_\lambda(t, u_m(t), w_m(t), z_m(t))dt = \int_0^T t^k F_\lambda(t, u_0(t), w_0(t), z_0(t))dt.$$

Therefore, we infer the existence of the following limit

$$\lim_{m\to\infty} \int_0^T t^k F_\lambda^* \left(t, -\frac{v_m'(t)}{t^k}, w_m(t), z_m(t)\right) dt.$$

Now we note that for all $u \in L^q(0,T)$, $m \in N$ and a.e. $t \in [0,T]$ one has

$$\begin{aligned} & -v_m'(t))u(t) - t^k F_\lambda(t, u(t), w_m(t), z_m(t)) \\ \leq\ & \sup_{r \in R} \left\{ -v_m'(t)r - t^k F_\lambda(t, r, w_m(t), z_m(t)) \right\} \\ =\ & t^k F_\lambda^* \left(t, -\frac{v_m'(t)}{t^k}, w_m(t), z_m(t)\right) \end{aligned}$$

and further

$$\begin{aligned}&\lim_{m\to\infty}\int_0^T \left(-v_m'(t))u(t) - t^k F_\lambda(t,u(t),w_m(t),z_m(t))\right) dt\\ \le\ &\lim_{m\to\infty}\int_0^T t^k F_\lambda^*\left(t,-\frac{v_m'(t)}{t^k},w_m(t),z_m(t)\right)dt.\end{aligned}$$

Combining (2.5), (2.6) and the previous inequality we derive

$$\begin{aligned}&\int_0^T (-v_0'(t))u(t) - t^k F_\lambda(t,u(t),w_0(t),z_0(t)))dt\\ \le\ &\lim_{m\to\infty}\int_0^T t^k F_\lambda^*\left(t,-\frac{v_m'(t)}{t^k},w_m(t),z_m(t)\right)dt\end{aligned}$$

for all $u \in L^q(0,T)$. Consequently

$$\begin{aligned}&\sup_{u\in L^q(0,T)}\left\{\int_0^T \left(-v_0'(t))u(t) - t^k F_\lambda(t,u(t),w_0(t),z_0(t))\right) dt\right\}\\ \le\ &\lim_{m\to\infty}\int_0^T t^k F_\lambda^*\left(t,-\frac{v_m'(t)}{t^k},w_m(t),z_m(t)\right)dt.\end{aligned}$$

Since

$$\begin{aligned}&\int_0^T t^k F_\lambda^*\left(t,-\frac{v_0'(t)}{t^k},w_0(t),z_0(t)\right)dt\\ =\ &\sup_{u\in L^q(0,T)}\left\{\int_0^T \left(-v_0'(t))u(t) - t^k F_\lambda(t,u(t),w_0(t),z_0(t))\right) dt\right\},\end{aligned}$$

we have

(2.7)
$$\int_0^T t^k F_\lambda^*\left(t,-\frac{v_0'(t)}{t^k},w_0(t),z_0(t)\right)dt \le \lim_{m\to\infty}\int_0^T t^k F_\lambda^*\left(t,-\frac{v_m'(t)}{t^k},w_m(t),z_m(t)\right)dt.$$

Taking into account (2.5), (2.6) and (2.7) we get (2.4).

Assertions (2.3) and (2.4) give

$$\frac{1}{q'(t^k a(t))^{\frac{q'}{q}}}\left|v_0(t)\right|^{q'} + \frac{a(t)t^k}{q}\left|u_0'(t)\right|^q - u_0'(t)v_0(t)) = 0 \text{ a.e in } (0,T)$$

and

$$\left(v_0'(t)u_0(t) + t^k F_\lambda^*\left(t,-\frac{v'(t)}{t^k}\right) + t^k F_\lambda(t,u_0(t),w_0(t),z_0(t))\right) = 0 \text{ a.e in } (0,T).$$

Consequently, by the properties of the Fenchel transform,

$$v_0(t) = t^k a(t)\left|u_0'(t)\right|^{q-2}u_0'(t) \text{ a.e. in } (0,T). \tag{2.8}$$

By (2.1),

$$-v_0'(t) = t^k\left(f_1(t,u_0(t),w_0(t)) + \lambda f_2(t,u_m(t),z_0(t))\right) \text{ a.e. in } (0,T). \tag{2.9}$$

Thus

$$-\left(t^k a(t)\left|u_0'(t)\right|^{q-2} u_0'(t)\right)' = t^k\left(f_1(t,u_0(t),w_0(t)) + \lambda f_2(t,u_m(t),z_0(t))\right) \tag{2.10}$$

a.e. in $(0,T)$. Note that $u_0(T)=0$, $0 \le u_0 \le d$ in $(0,T]$ and u_0 is continuous in $(0,T]$. By (2.8), the continuity of v_0 in $[0,T]$ implies that $u_0 \in C^1((0,T])$. Further (2.8), (2.9) and assumption (f2p) imply $t^k a(t)\left|u_0'(t)\right|^{q-2} u_0' \in A([0,T])$. Now it suffices to show that $u'(t) < 0$ for $t \in [0,T]$ and u_0 is continuous at 0. From (2.10) we have the estimates (as in the proof of Lemma 1.2)

$$\begin{aligned}
|u_0'(t)|^{q-1} &= \frac{1}{a(t)t^k}\int_0^t r^k f_1(r,u_0(r)) + \lambda f_2(r,u_0(r))dr \\
&\le \frac{1}{a(t)t^k}\left(\frac{1}{qk+1}\right)^{1/q} t^{k+1/q}\left(\int_0^T (\overline{f_1}(l,d) + \lambda\overline{f_2}(t,d))^{q'} dl\right)^{1/q'} \\
&\le \frac{1}{a_{\min}}\left(\frac{1}{qk+1}\right)^{1/q}\left(\int_0^T (\overline{f_1}(l,d) + \lambda\overline{f_2}(t,d))^{q'} dl\right)^{1/q'} t^{1/q}.
\end{aligned}$$

Finally

$$\lim_{t\to 0^+} u_0'(t) = 0 = u_0'(0).$$

Now (2.10) and Lemma 1.1 lead to the conclusion that $u_0'(t) < 0$ for $t \in (0,T)$. Thus $u_0 \in U$. □

Example 2.2. For $\lambda \in (6.935, 8.366)$ and all $(w,z) \in L^{p_1}(0,T) \times L^{p_2}(0,T)$, with $p_1, p_2 > 2$, the BVP

$$\left\{\begin{array}{c}
-\left(\left(\frac{1}{1+t^2}\left|u'(t)\right|^2 u'(t)\right)' + \frac{k}{(1+t^2)t}\left|u'(t)\right|^2 u'(t)\right) \\
= \frac{1}{810\sqrt{t}}\left(-u^4(t)\left(1+arctg^2 w(t)\right) + \lambda\left(u^2(t)+1\right)\left(1+\sin^2 z(t)\right)\right) \; a.e. \text{ in } (0,3), \\
u'(0) = 0 \text{ and } u(3) = 0.
\end{array}\right. \tag{2.11}$$

possesses at least one positive solution in the set $U := \{u \in \widetilde{U} \ ; \ u(t) \le 1 \ \textit{for all} \ t \in [0,3]\}$, with

$$\begin{aligned}
\widetilde{U} \;=\; &\Big\{u \in C^1([0,T]) : u(3) = 0 \text{ and } u'(0) = 0 \text{ and } u'(t) < 0 \\
&\text{for } t \in [0,3] \text{ and } \frac{t^k}{1+t^2}\left|u'\right|^2 u' \in A([0,T])\Big\}.
\end{aligned} \tag{2.12}$$

Moreover if for each $m \in N$, $u_m \in U$ denotes the solution of (2.11) for (w_m, z_m) and if $(w_m, z_m)_{m\in N}$ tends a.e. in $[0,T]$ to (w_0, z_0), then the sequence of solutions $(u_m)_{m\in N}$ tends uniformly (up to a subsequence) to a certain $u_0 \in U$ being a solution of (2.11) for parameters (w_0, z_0).

Proof. We consider (1.5) with $T = 3$, $q = 4$, $a(t) = \frac{1}{1+t^2}$ and

$$f_1(t,u,x) = -\frac{1}{810}\frac{1+arctg^2 x}{\sqrt{t}}u^4;$$
$$f_2(t,u,y) = \frac{1}{810}\frac{1+\sin^2 y}{\sqrt{t}}\left(u^2+1\right).$$

We show that all the ssumptions of Theorem 2.1 are satisfied in this case. First we look for λ such that

$$\begin{aligned} f(\ \ t,u,x,y) \ \ : &= f_1(t,u,x) + \lambda f_2(\ \ t,u,y) \\ &= \frac{1}{810\sqrt{t}}\left(-u^4\left(1+arctg^2 x\right) + \lambda\left(u^2+1\right)\left(1+\sin^2 y\right)\right) \end{aligned}$$

is increasing in $[0,1]$. Note

$$f'_u(t,u,x,y) = \frac{1}{810\sqrt{t}}\left(-4u^3\left(1+arctg^2 x\right) + 2\lambda u\left(1+\sin^2 y\right)\right)$$

and further

$$f'_u(t,u,x,y) = 0 \Leftrightarrow -4u^3\left(1+arctg^2 x\right) + 2\lambda u\left(1+\sin^2 y\right) = 0$$

which gives

$$u_0 = 0 \text{ or } u_1 = \sqrt{\frac{\lambda}{2}}\sqrt{\frac{1+\sin^2 y}{1+arctg^2 x}} \text{ or } u_2 = -\sqrt{\frac{\lambda}{2}}\sqrt{\frac{1+\sin^2 y}{1+arctg^2 x}}.$$

Thus for a.a. $t \in (0,T)$ and all $x, y \in R$

$$f'_u(t,u,x,y) > 0 \text{ for } u \in (-\infty, u_2) \cup (0, u_1)$$
$$f'_u(t,u,x,y) < 0 \text{ for } u \in (u_2, 0)$$

which implies that for a.a. $t \in (0,T)$ and $x, y \in R$ the function $f(t,\cdot,x,y)$ is increasing for $u \in (0,u_1)$. Moreover , since $f(t,0,x,y) = \frac{\lambda}{810\sqrt{t}}\left(1+\sin^2 y\right) > 0$, one sees that $f(t,u,x,y) > 0$ for $u \in (0,u_1)$. Thus we obtain

$$1 \le \sqrt{\frac{\lambda}{2}}\sqrt{\frac{1+\sin^2 y}{1+arctg^2 x}}$$

for a.a. $t \in (0,T)$ and all $x, y \in R$. Note that

$$\sqrt{\frac{1}{1+\pi^2/4}} \le \sqrt{\frac{1+\sin^2 y}{1+arctg^2 x}} \le \sqrt{2}$$

for all $x, y \in R$. We take λ such that

$$1 \le \sqrt{\frac{\lambda}{2}}\sqrt{\frac{1}{1+\pi^2/4}},$$

namely

$$\lambda \ge \frac{4+\pi^2}{2} \approx 6.9348. \tag{2.13}$$

We also look for λ such that

(2.14)
$$\int_0^3 \left(\frac{1}{a(s)s^k} \int_0^s r^k \frac{1}{810\sqrt{r}} \left(-d^4 \left(1 + arctg^2 x\right) + \lambda \left(d^2+1\right)\left(1+\sin^2 y\right)\right) dr \right)^{\frac{1}{3}} ds \le d$$

with $d = 1$. It is easy to see that

$$\begin{aligned} & \int_0^3 \left(\frac{1}{a(s)s^k} \int_0^s r^k \frac{1}{810\sqrt{r}} \left(-\left(1 + arctg^2 x\right) + 2\lambda \left(1+\sin^2 y\right)\right) dr \right)^{\frac{1}{3}} ds \\ \le & \int_0^3 \left(\frac{1}{810} \frac{1}{1+s^2} \int_0^s \frac{1}{\sqrt{r}} \left(-\left(1+\frac{\pi^2}{4}\right) + 4\lambda \right) dr \right)^{\frac{1}{3}} ds \\ = & \int_0^3 \left(\frac{1}{810} \frac{\left(4\lambda - \left(1+\frac{\pi^2}{4}\right)\right)}{1+s^2} \int_0^s \frac{dr}{\sqrt{r}} \right)^{\frac{1}{3}} ds \\ = & \sqrt[3]{\frac{1}{810}\left(4\lambda - \left(1+\frac{\pi^2}{4}\right)\right)} \int_0^3 \left(\frac{2\sqrt{s}}{1+s^2} \right)^{\frac{1}{3}} ds \\ \le & \; 2.7 \sqrt[3]{\frac{1}{810}\left(4\lambda - \left(1+\frac{\pi^2}{4}\right)\right)} \le \sqrt[3]{\frac{1}{30}\left(4\lambda - \left(1+\frac{\pi^2}{4}\right)\right)}. \end{aligned}$$

Therefore (2.14) holds if

$$\sqrt[3]{\frac{1}{30}\left(4\lambda - \left(1+\frac{\pi^2}{4}\right)\right)} \le 1$$

which is equivalent to

(2.15)
$$\lambda \le \frac{31}{4} + \frac{\pi^2}{16} \approx 8.3669.$$

Summarizing, for λ satisfying (2.13) and (2.15) all the assumtions of Theorem 2.1 hold. $\square$

REFERENCES

[1] D. Bonheure & J.M.Gomes & L.Sanchez, *Positive solutions of a second-order singular ordinary differential equation*, Nonlinear Analysis 61 (2005) 1383–1399.

[2] M.Grossi, D.Passaseo, *Nonlinear elliptic Dirichlet problems in exterior domains: the role of geometry and topology of the domain*, Comm. Appl. Nonlinear Anal. 2(2) (1995) 1–31.

[3] R. Molle and D. Passaseo, *Multiple solutions of nonlinear elliptic Dirichlet problems in exterior domains*, Nonlinear Analysis 39 (2000) 447–462.

[4] H. Lü, D.O'Regan and R.P. Agarwal, *An approximation approach to eigenvalue intervals for singular boundary value problems with sign changing nonlinearities,* Mathematical Inequalities and Applications, **11** (2007), 81–98.

[5] H. Lü, D.O'Regan and R.P. Agarwal, *Existence to singular boundary value problems with sign changing nonlinearities using an approximation methods approach*, Applications of Mathematics, **52** (2007), 117–135.

[6] D. O'Regan and A.Orpel, *Eigenvalue problems for singular odes,* Glasgow Mathematical Journal, in press.

[7] A. Nowakowski and A. Rogowski, *Multiple positive solutions for a nonlinear Dirichlet problem with non-convex vector-valued response*, Proc. Royal Soc. Edinburgh **135**A, (2005), 105–117.

[8] A. Orpel, *On the existence of positive radial solutions for a certain class of elliptic*, J. Math. Anal. Appl. **299** (2004), 690–702.

[9] A. Orpel, *On the existence of bounded positive solutions for a class of singular BVP*, Nonlinear Analysis, 69, (2008), 1389–1395.