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OPTIMAL INVESTMENT AND PROPORTIONAL REINSURANCE UNDER NO SHORT-SELLING AND NO BORROWING

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ABSTRACT. Insurance companies resort to investment and reinsurance, among other options, to manage their reseerves. This article addresses the problem of optimal investment and reinsurance when no short-selling and no borrowing allowed. More specifically, we assume that the risk process of the insurance company is a compound Poisson process perturbed by a standard Brownian motion and that the risk can be reduced through a proportional reinsurance. In addition, the surplus can be invested in the financial market such that the portfolio will consist, for simplicity, of one risky asset and one risk-free asset. Our goal is to find the optimal investment and reinsurance policy which can maximize the expected exponential utility of the terminal wealth. In the case of no short-selling, we find the closed form of value function as well as the optimal investment-reinsurance policy. In the case when neither short-selling nor borrowing allowed, the resulting HJB equation is difficult to solve analytically, and hence we provide a numerical solution through Markov chain approximation techniques.

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1. INTRODUCTION

In this age of fierce competition among businesses, insurance companies look for ways of increasing their reserve and minimize their risk. To increase the reserve, the insurance companies resort to various investment strategies. Reinsurance is one of the ways insurance companies effectively reduce their exposure to loss.

The optimal control problem of investment and of reinsurance have attracted a lot of attention in the past few years. In applying stochastic control to maximize the expected utility of the terminal wealth, the articles [2] and [20] model the risk process with diffusion approximation and jump-diffusion processes, respectively; and, they offer the optimal investment strategy (without considering reinsurance). Liang [12] studies the optimal problem of investment and proportional reinsurance for the jump-diffusion risk processes, and obtain the closed form expressions for the strategy and the value function. Assuming that the risky asset follows an independent jump-diffusion process, the articles [9] and [19] provide the optimal investment and proportional reinsurance policies to maximize the expected exponential and other utility functions of the terminal wealth. In [13] the authors combine proportional and excess of loss reinsurance to maximize the expected exponential utility of the terminal wealth. Interested reader can find in [18, 17, 14] another kind of value function which minimizes the ruin probability.

The constraints on the controls of many of the above models are not natural, and some of them are not even legally permitted. Since countries such as China impose restrictions on short selling, it is important to study the associated optimal control problems without short selling opportunity. Under the assumption of no shorting, Bai and Guo [1] considered the optimization problems of maximizing the expected exponential utility of terminal wealth and of minimizing the probability of ruin for a diffusion approximation risk process. Cao and Wan [3], under the constraints of short-selling and borrowing, also model the risk process with the diffusion approximation and partly solve the optimal control problem. Irgens and Paulsen [9] briefly discuss the constraints on the risky portfolio and reinsurance quantitatively and qualitatively, and show that some constraints turn what is originally a fairly simple problem into a very difficult one resulting in applying numerical methods. In the jump diffusion process model, Liu and Zhang [15] considers the optimal problem of investment and excess of loss(XL) reinsurance under no short selling. Also in this current work, we drop the short selling possibility. Borrowing is also a risky strategy for an insurance company, and we do not allow borrowing in our work.

In this paper, we consider the stochastic optimal control problems without the possibilities of short selling and borrowing, and solve the problems using analytical and numerical methods. Our model involves more realisitic jump-diffusion risk process rather than the diffusion approximation. More specifically, we assume that the risk process is a compound Poisson process perturbed by a standard Brownian motion, and that the risk can be reduced through proportional reinsurance. In addition, the surplus is invested in a financial market consisting, for simplicity, of one risky asset and one risk-free asset. Our goal is to find the optimal investment and reinsurance policy which maximizes the expected exponential utility of the terminal wealth.

The rest of this article is organized as follows. We formulate the model assumptions in Section 2. Restricting short selling, Section 3 solves the corresponding Hamilton-Jacobi-Bellman equation and find the closed form expression for the value function as well as the optimal investment and reinsurance policy. With no short selling and no borrowing, the associated HJB equation becomes difficult to solve analytically. So in Section 4, we present the numerical solution through the markov chain approximation techniques. Section 5 concludes the article.

2. THE MODEL AND THE HJB EQUATION

2.1. **The model.** We model the reserve or suplus process for an insurance company in terms of a diffusion perturbed compound Poisson process. In order to write down the dynamic equation for the reserve process, we first introduce the needed notations.
• $(\Omega, \mathcal{F}, \mathcal{F}_t, P)$ is a filtered complete probability space supporting all our random elements and satisfying usual regularity condition.
• $\{W_t^{(1)},\ t \geq 0\}$ is a standard Brownian motion (adapted to $\mathcal{F}_t,\ t \geq 0$). This process arises out of the uncertainty associated with the insurance market and/or the economic environment. The uncertainty is not necessarily related to the claims. Therefore, we only consider the case where $\beta dW_t^{(1)}$ is not affected by reinsurance at all [13], β is a constant.
• Let Y_i be IID random variable with distribution function Φ and density function φ. These random variables represent the claims by the customers from the insurance company.
• Since proportional reinsurance is involved in our study, the insurer and the reinsurer will pay claimants the amounts qY_i and $(1-q)Y_i$, respectively. Here, the proportion $0 \leq q \leq 1$ is called the *risk exposure of the insurer.* We assume that the reinsurance proportion q can change continuously in time t such that $q(t)$ is a predictable process w.r.t $\mathcal{F}_t$.
• $N_t,\ t \geq 0$, is a Poisson process with intensity λ, and is independent of the claim amounts Y_i and of the Brownian motion $W_t^{(1)},\ t \geq 0$. The total claim up to time t is given by the compound Poisson process

$$S(t) := \sum_{i=1}^{N(t)} Y_i.$$

• Now the surplus process $R(t),\ t \geq 0$, satisfies the equation

$$dR(t) = cdt + \beta dW_t^{(1)} - dS(t), \tag{1}$$

where $c > 0$ is the constant premium income rate. The insurer is partially transferring risk, via reinsurance, to the reinsurer. Therefore, the insurer should pay the reinsurer in the form of an upfront reinsurance premium.
• In our work, we assume that the premium is calculated according to the commonly used variance principle. That is, the reinsurance premium is given by

$$(1-q)\lambda\mu + \alpha(1-q)^2\lambda\mu_2,$$

where $\alpha > 0,\ \mu = EY_i,$ and $\mu_2 = EY_i^2$.

• The insurer can invest all of the surplus in the financial market, in which the standard assumptions of continuous-time financial models hold:

1) continuous trading is allowed;

2) no transaction cost or tax is involved in trading; and

3) all assets are infinitely divisible.

• For simplicity, we assume that the financial market consists of one risk-free asset(savings account) whose price at time t is denoted by B_t and one risky asset(stock) whose price at time t is denoted by P_t. Let the investment fraction of the two assets at time t be $1 - b(t)$ and $b(t)$ respectively, where $b(t)$ is assumed to be predictable. The price of risk-free asset follows:

$$dB(t) = r_0 B(t)dt,$$

where, $r_0 \geq 0$ is the risk-free interest rate. The price of the risky asset is given by:

$$dP(t) = r_1 P(t)dt + \sigma P(t)dW_t^{(2)},$$

where, $r_1 > r_0,\ \sigma$ are positive constants that represent the expected instantaneous rate and the volatility of return on the risky asset, respectively, and $\{W_t^{(2)}, t > 0\}$ is another standard Brownian motion defined on the same probability space $(\Omega, \mathcal{F}, \mathbb{P})$. We assume that the joint distribution of the two Brownian motions that we use is bivariate normal, and we denote their correlation coefficient by ρ, i.e., $E[W_t^{(1)} W_t^{(2)}] = \rho t$.

• Combining the proportional reinsurance and investment, the dynamics of the surplus process of the insurer becomes:

$$\begin{aligned} dX_t =& cdt - [(1 - q(t))\lambda\mu + \alpha(1 - q(t))^2\lambda\mu_2]dt + \beta dW_t^{(1)} - q(t)dS_t \\ &+ r_1 b(t)X_t dt + \sigma b(t)X_t dW_t^{(2)} + (1 - b(t))r_0 X_t dt \\ =& [c - (1 - q(t))\lambda\mu - \alpha(1 - q(t))^2\lambda\mu_2 + r_1 b(t)X_t + (1 - b(t))r_0 X_t]dt \\ &+ \beta dW_t^{(1)} + \sigma b(t)X_t dW_t^{(2)} - q(t)dS_t, \end{aligned} \tag{2}$$

the initial wealth is $X_0 = x$. In the above equation, $(b(t), q(t))$ is the predictable control process and we let $\mathbb{U}$ to denote the set of all the admissible controls.

• The no-shorting constraint means that $b(t) \geq 0$.

• The no-borrowing constraint means that $b(t) \leq 1$.

• Under the assumption that the insurer's objective is maximizing the exponential utility of the terminal wealth, say at time T, the value function has the form as follows:

$$V(t, x) = \sup_{(q,b)\in\mathbb{U}} E[u(X_T^{q,b}) | X_t^{q,b} = x]. \tag{3}$$

where, $u(x)$ is the exponential utility function:

$$u(x) = c_0 - \frac{\delta}{\gamma} e^{-\gamma x}. \tag{4}$$

where, δ and $\gamma > 0$ are positive constants.

2.2. **HJB equation.** The infinitesimal generator of the jump diffusion given by Equation 2 is obtained by applying Itô's Lemma to a $C^{1,2}$ function $g(t, x)$, [4]. That is, the infinitesimal generator of the jump process X_t is given by

$$\begin{aligned}\mathcal{A}^{q,b} g(t,x) =& g_t + [c - (1-q)\lambda\mu - \alpha(1-q)^2\lambda\mu_2 + r_1 bx + (1-b)r_0 x]g_x \\ &+ \frac{1}{2}[\beta^2 + \sigma^2 b^2 x^2 + 2\rho\sigma b\beta x]g_{xx} + \lambda E[g(t, x - qY) - g(t,x)]\end{aligned}$$

where, g_x, g_t and g_{xx} denote the first order partial derivative with respect to x, the first order partial derivative with respect to t, and the second order partial derivative with respect to x, respectively.

If the value function V is smooth enough, then appealing to the theory of dynamic programming, [7], we note that $V(t, x)$ satisfies the Hamilton-Jacobi-Bellman equation:

$$\sup_{(q,b)\in[0,1]\times\mathcal{B}} \mathcal{A}^{q,b} V(t,x) = 0, \qquad V(T,x) = u(x), \tag{5}$$

where
(a) $\mathcal{B} = (-\infty, +\infty)$ in the case when both short-selling and borrowing are allowed,
(b) $\mathcal{B} = [0, 1]$ when there is neither short-selling nor borrowing, and
(c) $\mathcal{B} = [0, +\infty)$ if borrowing but not short-selling is allowed.

The standard procedure of verification can be used to prove that the classical solution of the HJB equation, when it exists, is the value function, see [20, 6, 7]. Also, the HJB Equation (5) takes the form

$$\begin{aligned}&V_t + \sup_q \Big\{[c - (1-q)\lambda\mu - \alpha(1-q)^2\lambda\mu_2]V_x + \lambda E[V(x - qY) - V(x)]\Big\} \\ &+ \sup_b \left\{(r_1 bx + (1-b)r_0 x)V_x + \frac{1}{2}(\beta^2 + \sigma^2 b^2 x^2 + 2\rho\sigma b\beta x)V_{xx}\right\} = 0.\end{aligned} \tag{6}$$

3. THE CASE OF NO SHORT-SELLING

In this section we solve the optimal control problem in the case when the short-selling in the investment is forbidden, that is, $b(t) \geq 0$. Differentiating the HJB Equation (6) with respect to b, and setting the derivative equal to zero, we get

$$b^0(t,x) = -\frac{r_1 - r_0}{\sigma^2 x} \cdot \frac{V_x}{V_{xx}} - \frac{\rho\beta}{\sigma x}.$$

Disallowing short-selling, the optimal fraction of risky asset is

$$b^*(t,x) = \begin{cases} b^0(t,x), & \text{if, } b^0(t,x) \geq 0; \\ 0, & \text{if } b^0(t,x) < 0. \end{cases}$$

Under the consideration of the above optimal fraction, we shall solve the HJB Equation (6). Toward this, we need to establish a couple of lemmas.

Lemma 1. *Let $\mathcal{O}_1 := \{(t,x) \in [0,T] \times \mathbb{R} : b^0(t,x) > 0\}$. Assume that $x \to V(t,x)$ is a concave and increasing function, and satisfies*

$$\begin{aligned} V_t &+ \left[c + r_0 x - \frac{\rho\beta}{\sigma}(r_1 - r_0)\right] V_x + \frac{1}{2}(1-\rho^2)\beta^2 V_{xx} - \frac{1}{2}\frac{(r_1-r_0)^2}{\sigma^2}\ \frac{V_x^2}{V_{xx}} \\ &+ \sup_{q\in[0,1]} \left\{(-(1-q)\lambda\mu - \alpha(1-q)^2\lambda\mu_2)V_x + \lambda E[V(x-qY) - V(x)]\right\} \\ &= 0, \qquad (t,x) \in \mathcal{O}_1, \qquad (7) \\ V(T,x) &= u(x). \end{aligned}$$

Then, V satisfies the HJB equation on $\mathcal{O}_1$.

Proof: On $\mathcal{O}_1$, the supremum over b of the HJB Equation (6) is attained at $b^0(t,x)$. Now, replace the $b(t)$ in the Equation(6) with $b^0(t,x)$ which will give us the Equation (7).

Lemma 2. *Let $\mathcal{O}_2 := \{(t,x) \in [0,T] \times \mathbb{R} : b^0(t,x) < 0\}$. Assume that $x \to V(t,x)$ is a concave and increasing function satisfying*

$$\begin{aligned} V_t &+ (c + r_0 x)V_x + \frac{1}{2}\beta^2 V_{xx} \\ &+ \sup_{q\in[0,1]} \left\{(-(1-q)\lambda\mu - \alpha(1-q)^2\lambda\mu_2)V_x + \lambda E[V(x-qY) - V(x)]\right\} \\ &= 0, \qquad (t,x) \in \mathcal{O}_2, \qquad (8) \\ V(T,x) &= u(x). \end{aligned}$$

Then V satisfies the HJB Equation on $\mathcal{O}_2$.

Proof: In $\mathcal{O}_2$, the supremum over b of the HJB Equation(6) is attained at 0. By replacing $b(t)$ in the Equation (6) with 0 we get the Equation (8).

We shall now proceed to solve the HJB Equation (6) for the maximal expected utility function of the terminal wealth, and find the optimal control policy of the reinsurance and the investment as well. We need to consider the following three cases separately.

1) $r_1 - r_0 < \rho\sigma\gamma\beta$,
2) $\rho\sigma\gamma\beta \leq r_1 - r_0 \leq e^{r_0 T}\rho\sigma\gamma\beta$,
3) $r_1 - r_0 > e^{r_0 T}\rho\sigma\gamma\beta$.

We solve the problem for the second case: $\rho\sigma\gamma\beta \leq r_1 - r_0 \leq e^{r_0 T}\rho\sigma\gamma\beta$, since the other two cases will follow from analogous arguments.

Let us begin with Equation (7) for $x \in \mathcal{O}_1$. Guided by [20], we shall fit a solution of the form

$$V(t,x) = c_0 - \frac{\delta}{\gamma}\exp\left\{-\gamma x e^{r_0(T-t)} - \frac{1}{2}\frac{(r_1-r_0)^2}{\sigma^2}(T-t) + h(T-t)\right\}, \tag{9}$$

with the boundary condition

$$V(T,x) = u(x) = c_0 - \frac{\delta}{\gamma}e^{-\gamma x} \tag{10}$$

(implying $h(0) = 0$). Let M_Y denote the moment generating function of the claim size r.v Y_i. For the above trial solution, we have

$$\begin{cases} V_t = [V(t,x) - c_0]\left[x r_0 \gamma e^{r_0(T-t)} + \frac{1}{2}\left(\frac{r_1-r_0}{\sigma}\right)^2 - h'(T-t)\right] \\ V_x = [V(t,x) - c_0]\left(-\gamma e^{r_0(T-t)}\right) \\ V_{xx} = [V(t,x) - c_0]\left(\gamma^2 e^{2r_0(T-t)}\right) \\ \lambda E[V(t,x-Y) - V(t,x)] = \lambda\left[V(t,x) - c_0][M_Y(\gamma e^{r_0(T-t)}) - 1\right] \end{cases} \tag{11}$$

where, recall that M_Y is the moment-generating function of the claim size variable Y_i. And now, it's easy to see, from Relations 11 and the definition of $b^0(t,x)$ above, that

$$b^0(t,x) = \frac{r_1 - r_0}{\sigma^2\gamma x e^{r_0(T-t)}} - \frac{\rho\beta}{\sigma x},$$

and also that

$$\mathcal{O}_1 = \{(t,x) : T - (\log(r_1 - r_0) - \log(\rho\sigma\gamma\beta))/r_0 < t < T\}.$$

Inserting now the Relations (11) into Equation (7), we get

$$\inf_{q\in[0,1]}\Big\{ - h'(T-t) - \left[c - \frac{\rho\beta}{\sigma}(r_1 - r_0) - (1-q)\lambda\mu - \alpha(1-q)^2\lambda\mu_2\right]\gamma e^{r_0(T-t)}$$
$$\frac{1}{2}(1-\rho^2)\beta^2\gamma^2 e^{2r_0(T-t)} + \lambda[M_Y(\gamma q e^{r_0(T-t)}) - 1]\Big\} = 0. \tag{12}$$

Differentiating the above equation(12) with respect to q, the stationary points $q^0(t,x)$ satisfy the following equation:

$$\mu + 2\alpha\mu_2(1-q) = M_Y'(\gamma q e^{r_0(T-t)}). \tag{13}$$

And the following lemma shows that the equation(13) has a unique solution in $[0,1]$.

Lemma 3. *The Equation* (13) *has a unique solution in* $[0,1]$.

Proof: The proof of this lemma can be found in [12].

As seen from Lemma 3, the optimal reinsurance proportion $q^*(t,x)$ is the unique solution $q^0(t,x)$. Replacing q in Equation (12) with $q^*(t,x)$, we get

$$h'(T-t) = -\left[c - \frac{\rho\beta}{\sigma}(r_1 - r_0) - (1-q^*(t,x))\lambda\mu - \alpha(1-q^*(t,x))^2\lambda\mu_2\right]\gamma e^{r_0(T-t)} + \frac{1}{2}(1-\rho^2)\beta^2\gamma^2 e^{2r_0(T-t)} + \lambda[M_Y(\gamma q^*(t,x)e^{r_0(T-t)}) - 1]. \tag{14}$$

Integrating this we get

$$h(T-t) = -\left[c - \frac{\rho\beta}{\sigma}(r_1 - r_0) - (1-q^*)\lambda\mu - \alpha(1-q^*)^2\lambda\mu_2\right]\gamma\frac{e^{r_0(T-t)}-1}{r_0} + \frac{1}{4}(1-\rho^2)\beta^2\gamma^2\frac{e^{2r_0(T-t)}-1}{r_0} + \int_0^{T-t}\lambda[M_Y(\gamma q^*(t,x)e^{r_0 u}) - 1]du \tag{15}$$

Similarly, for $(t,x) \in \mathcal{O}_2 = \{(t,x) : 0 < t < T - (\log(r_1 - r_0) - \log(\rho\sigma\gamma\beta))/r_0\}$, it follows from Lemma 2, that the value function has the form:

$$V(t,x) = c_0 - \frac{\delta}{\gamma}\exp\left\{-\gamma x e^{r_0(T-t)} + g(T-t)\right\}, \tag{16}$$

where,

$$\begin{aligned} g(T-t) &= -\left[c - (1-q_t^*)\lambda\mu - \alpha(1-q_t^*)^2\lambda\mu_2\right]\gamma\frac{e^{r_0(T-t)}-1}{r_0} \\ &+ \frac{1}{4}\beta^2\gamma^2\frac{e^{2r_0(T-t)}-1}{r_0} + \int_0^{T-t}\lambda[M_Y(\gamma q_t^* e^{r_0 u}) - 1]du + k. \end{aligned} \tag{17}$$

in which k is an undetermined constant. By the continuity in t of the value function, we notice that k is given by

$$k = \frac{3(r_1-r_0)^2}{4\sigma^2 r_0} + \frac{\rho^2\beta^2\gamma^2}{4r_0} - \frac{\rho\beta\gamma}{\sigma r_0}(r_1 - r_0) - \frac{(r_1-r_0)^2}{2\sigma^2 r_0}\log\frac{r_1 - r_0}{\rho\sigma\gamma\beta}.$$

For $(t,x) \in \mathcal{O}_2$, $b^*(t,x) = 0$ and $q^*(t,x)$ is also the unique solution of the equation(13) in $[0,1]$.

The above discussions establish the following results.

Theorem 4. *Assume that*

$$\rho\sigma\gamma\beta \le r_1 - r_0 \le e^{r_0 T}\rho\sigma\gamma\beta.$$

In the case of

$$0 < t < T - (\log(r_1 - r_0) - \log(\rho\sigma\gamma\beta))/r_0,$$

the value function and the optimal reinsurance-investment policy are given by the following.

1. *The value function is*

$$V(t,x) = c_0 - \frac{\delta}{\gamma}\exp\left\{-\gamma x e^{r_0(T-t)} + g(T-t)\right\}, \tag{18}$$

where, $g(T-t)$ *is given by*

$$g(T-t) = -\left[c - (1-q_t^*)\lambda\mu - \alpha(1-q_t^*)^2\lambda\mu_2\right]\gamma\frac{e^{r_0(T-t)}-1}{r_0} + \frac{1}{4}\beta^2\gamma^2\frac{e^{2r_0(T-t)}-1}{r_0} + \int_0^{T-t}\lambda[M_Y(\gamma q_t^* e^{r_0 u}) - 1]du + k,$$

in which, under continuity,

$$k = \frac{3(r_1-r_0)^2}{4\sigma^2 r_0} + \frac{\rho^2\beta^2\gamma^2}{4r_0} - \frac{\rho\beta\gamma}{\sigma r_0}(r_1-r_0) - \frac{(r_1-r_0)^2}{2\sigma^2 r_0}\log\frac{r_1-r_0}{\rho\sigma\gamma\beta},$$

and M_Y *is the moment-generating function of the claim size variable* Y_i*.*

2. *The optimal reinsurance proportion* q^**, is the unique solution of the following Equation* (19) *in* $[0,1]$*:*

$$\mu + 2\alpha\mu_2(1-q) = M_Y'(\gamma q e^{r_0(T-t)}) \tag{19}$$

3. *The optimal investment fraction to risky asset is* $b^* = 0$*.*

Theorem 5. *Assume that*

$$\rho\sigma\gamma\beta \le r_1 - r_0 \le e^{r_0 T}\rho\sigma\gamma\beta.$$

In the case

$$T - \left(\log(r_1-r_0) - \log(\rho\sigma\gamma\beta)\right)/r_0 < t < T,$$

we have the following:

1. *The value function is given by*

$$V(t,x) = c_0 - \frac{\delta}{\gamma}\exp\Big\{-\gamma x e^{r_0(T-t)} - \frac{1}{2}\frac{(r_1-r_0)^2}{\sigma^2}(T-t) + h(T-t)\Big\}, \tag{20}$$

where $h(T-t)$ *is given by :*

$$h(T-t) = -\left[c - \frac{\rho\beta}{\sigma}(r_1-r_0) - (1-q^*(t,x))\lambda\mu - \alpha(1-q^*)^2\lambda\mu_2\right]\gamma\frac{e^{r_0(T-t)}-1}{r_0} + \frac{1}{4}(1-\rho^2)\beta^2\gamma^2\frac{e^{2r_0(T-t)}-1}{r_0} + \int_0^{T-t}\lambda[M_Y(\gamma q^* e^{r_0 u}) - 1]du.$$

2. *The optimal reinsurance proportion* q^* *is the unique solution of the Equation* (19) *in* $[0,1]$.
3. *The optimal investment fraction to risky asset is*

$$b^* = \frac{r_1-r_0}{\sigma^2\gamma x e^{r_0(T-t)}} - \frac{\rho\beta}{\sigma x}$$

.

Theorem 6. *Assume that*

$$r_1 - r_0 < \rho\sigma\gamma\beta.$$

Then,

1. *The value function has the same form as in Equation* (18) *with* $k = 0$.
2. *The optimal reinsurance proportion* q^* *is the unique solution of equation* (19) *in* $[0, 1]$.
3. *The optimal investment fraction to risky asset is* $b^* = 0$.

Theorem 7. *Assume that*

$$r_1 - r_0 > e^{r_0 T}\rho\sigma\gamma\beta.$$

Then,

1. *The value function has the same form as in Equation* (20).
2. *The optimal reinsurance proportion* q^* *is the unique solution of the Equation* (19) *in* $[0, 1]$.
3. *The optimal investment fraction to risky asset is given by*

$$b^* = \frac{r_1 - r_0}{\sigma^2 \gamma x e^{r_0(T-t)}} - \frac{\rho\beta}{\sigma x}.$$

Remark 1.

1. For the results of the optimal investment-XL reinsurance problem without short-selling constraint, the readers can refer to [12, 20], although the model assumptions are not exactly the same. We can see from our Theorem **??** that the consideration of short- selling constraint has changed the form of the optimal investment fraction to the risky asset. The optimal investment strategy and the value function will change also for different parametric values.
2. The term $(r_1 - r_0)/\sigma$ can be regarded as a measure of market price of financial risk. The short-selling is highly risky, and the resulting loss can become infinite.
3. When the price goes down, one needs to decrease the possession of the risky asset. The short-selling constraint makes the optimal investment fraction to be zero.
4. When the price is high, there is no need for short-selling, and the optimal investment fraction is the same as that in the case of without short-selling constraint.

Remark 2. Under the assumption $\rho\sigma\gamma\beta \leq r_1 - r_0 \leq e^{r_0 T}\rho\sigma\gamma\beta$, the optimal investment fraction to risky asset is

$$b^* = \frac{r_1 - r_0}{\sigma^2 \gamma x e^{r_0(T-t)}} - \frac{\rho\beta}{\sigma x}.$$

This becomes greater than 1 for $x < \frac{r_1 - r_0}{\sigma^2 \gamma e^{r_0(T-t)}} - \frac{\rho\beta}{\sigma}$. In other words, the optimal results under no short-selling are not optimal under the short-selling and borrowing constraints any more. In order to find the optimal policy under short-selling and borrowing constraints, the HJB equation takes the form

$$V_t + rxV_x + (ax^2 + bx + c)V_{xx} + \sup_q \big(q^2 V_x + E[V(x - qY) - V(x)]\big) = 0.$$

When we try a solution in the general form of

$$V(t,x) = c_0 - \frac{\delta}{\gamma}\exp\{f_0(t) + f_1(t)x + f_2(t)x^2\},$$

there is no way to determine the function $f_0(t)$, $f_1(t)$ and $f_2(t)$. This happens because of the variable coefficients arising in the integro-partial differential equation making it hard to separate the variables t and x.

4. THE CASE WITH NEITHER SHORT-SELLING NOR BORROWING

Under the constraints of no short selling and no borrowing, the fraction of risky asset is limited to $[0, 1]$. In this case, the corresponding HJB equation turns out to be difficult to solve analytically. Therefore, as an alternative to solving the HJB equation, we design a numerical algorithm for this problem by adopting and modifying the Markov chain approximation method. Whereas we did not consider in [15] the optimal control problem of investment-XL reinsurance under short-selling and borrowing constraints, that problem can also be solved through the numerical algorithm described below. The Markov chain approximation type numerical method is developed by Kushner *et al*, [10, 11]. The references [16] and [8] describe the Markov chain approximation method in more detail for diffusions and jump-diffusions, respectively. By the Markov chain approximation method, the numerical solution of our problem can be derived through the following steps:

Step 1. Discretization of the problem.

The first step is to discretize the continuous state space and the continuous time set. In our problem, the state variable is the wealth of the insurer X_t described by the Equation (2) *supra*. We approximate X_t by a discrete-time, discrete-state Markov process $\xi = (\xi_t^{h,\tau})$ which is locally consistent with X_t. The discretized time set is defined by

$$\{0, \tau, 2\tau, \cdots, (T/\tau)\tau\},$$

where T/τ is integer. We discretize the state space into

$$R^h = \{0, h, 2h, \cdots, Ih\},$$

where $\bar{x} \equiv Ih$ is an artificial upper bound which tends to infinity as $h \to 0$. Having setup the discrete time set and the discrete state space for the Markov chain approximation, we shall now proceed to describe the transition probabilities of the Markov chain $\xi = (\xi_t^{h,\tau})$.

Step 2. Construction of the Markov chain approximation.

Because of the separability of the diffusion part and the jump part in the problem, we can treat these two component separately.

(i) **The diffusion part.**

Ignoring, for the moment, the jump part in the wealth process, we continue to denote the remaining part by X_t.

$$\begin{aligned} dX_t &= [c-(1-q)\lambda\mu-\alpha(1-q)^2\lambda\mu_2+r_1bX_t+(1-b)r_0X_t]dt \\ &\quad +\beta dW_t^{(1)}+\sigma bX_t dW_t^{(2)}. \end{aligned}$$

According to the theory of dynamic programming [7], we have the following Hamilton-Jacobi-Bellman equation:

$$V_t+\sup_{(q,b)}\Bigg[\Big(c-(1-q)\lambda\mu-\alpha(1-q)^2\lambda\mu_2+r_1bx+(1-b)r_0x\Big)V_x + \frac{1}{2}(\beta^2+\sigma^2b^2x^2+2\rho\sigma b\beta x)V_{xx}\Bigg]=0.$$

In order to write down the transition probabilities, which are locally consistent with the transition mechanism of X_t, we follow the method in [16] and consider the following equation, which corresponds to the HJB Equation (6):

$$V_t+f(x,q,b)V_x+\frac{1}{2}g(x,q,b)V_{xx}=0. \tag{21}$$

Here,

$$\begin{aligned} f(x,q,b) &= c-(1-q)\lambda\mu-\alpha(1-q)^2\lambda\mu_2+r_1bx+(1-b)r_0x, \\ g(x,q,b) &= \beta^2+\sigma^2b^2x^2+2\rho\sigma b\beta x. \end{aligned}$$

We discretize the Equation (21) using the finite difference method.

$$\begin{aligned} V_t &\to \frac{V(x,t)-V(x,t-\tau)}{\tau} \\ V_x &\to \frac{V(x+h,t)-V(x,t)}{h}, \qquad f>0 \\ V_x &\to \frac{V(x,t)-V(x-h,t)}{h}, \qquad f<0 \\ V_{xx} &\to \frac{V(x+h,t)-2V(x,t)+V(x-h,t)}{h^2} \end{aligned}$$

where, $x = kh, t = n\tau(k, n$ are positive integers). Substitute the above discretizations into Equation (21) and simplify to get:

$$V(x,t-\tau) = V(x,t)\left(1 - \frac{|f(x,q,b)|\delta}{h} - \frac{g(x,q,b)\delta}{h^2}\right) + V(x+h,t)\left(\frac{f^+(x,q,b)\delta}{h} + \frac{g(x,q,b)\delta}{2h^2}\right) + V(x-h,t)\left(\frac{f^-(x,q,b)\delta}{h} + \frac{g(x,q,b)\delta}{2h^2}\right)$$

From this the transition probabilities can be defined as

$$\begin{aligned} p^{(D)}(x,x|q,b,t) &= 1 - \frac{|f(x,q,b)|\delta}{h} - \frac{g(x,q,b)\delta}{h^2}, \\ p^{(D)}(x,x+h|q,b,t) &= \frac{f^+(x,q,b)\delta}{h} + \frac{g(x,q,b)\delta}{2h^2}, \\ p^{(D)}(x,x-h|q,b,t) &= \frac{f^-(x,q,b)\delta}{h} + \frac{g(x,q,b)\delta}{2h^2}, \\ p^{(D)}(x,y|q,b,t) &= 0, \qquad y \neq x, x \pm h, \end{aligned}$$

for $x \in \{h, 2h, \ldots, (I-1)h\}$ and

$$\begin{aligned} p^{(D)}(\bar{x},\bar{x}-h|q,b,t) &= \frac{f^-(x,q,b)\delta}{h} + \frac{g(x,q,b)\delta}{2h^2}, \\ p^{(D)}(\bar{x},\bar{x}|q,b,t) &= 1 - p^{(D)}(\bar{x},\bar{x}-h|q,b,t), \\ p^{(D)}(\bar{x},y|q,b,t) &= 0 \qquad y \neq \bar{x}, \bar{x}-h, \end{aligned}$$

and

$$\begin{aligned} p^{(D)}(0,0|q,b,t) &= 1, \\ p^{(D)}(0,y|q,b,t) &= 0, \qquad y \neq 0. \end{aligned}$$

(ii) **The case that includs the jump part.**

In this part, we extend the above results to the jump-diffusion process. In our problem, the jump process is the compound Poisson process S_t whose jump intensity is $\lambda(t)$ and jump-amplitude function is $q(t,x)Y$. From the properties of the compound Poisson process, the probability Poisson-jump in time-steps of τ can be written as

$$p^{(J)} = \begin{cases} 1 - \lambda\tau + o(\tau), & 0 \ jump, \\ \lambda\tau + o(\tau), & 1 \ jump, \\ o(\tau), & 2 \ jumps. \end{cases}$$

However, the treatment of jump case is much more complicated than that for diffusion. The diffusion case the dependence is only local, depending on only the nearest neighbor (or, nodes). In the jump process case, the jump behavior is globally dependent on nodes that may be remote from the current node [8]. It's easy to see, in our problem, that the post-jump stage $y = x + q(t,x)Y$ is uniquely invertible with Y as a function of y given x. However, in order to get a positive probability of jumping to the post-jump state, it is necessary to have a set-target $\mathcal{S}(X_\ell)$ rather than a point-target $y = X_\ell$ such that $\mathcal{S}(X_\ell)$ form a partition of the state domain $[0, \bar{x}]$. Under the condition that the current node is X_j and post-jump point is in $\mathcal{S}(X_\ell)$, the probability of corresponding jump-amplitude is:

$$\begin{aligned}
&\text{Prob}[y = x + q(t,x)Y \in \mathcal{S}(X_\ell) | x = X_j, y \in \mathcal{S}(X_\ell)] \\
&= \begin{cases} \text{Prob}[Y \in (\mathcal{S}(X_\ell) - x)/q(t,x) | x = X_j], & \text{if } q(t,x) \neq 0, \\ 1, & \text{if } j = \ell,\ q(t,x) = 0, \\ 0, & \text{if } j \neq \ell,\ q(t,x) = 0 \end{cases} \\
&= \bar{\Phi}(X_j, X_\ell, t).
\end{aligned}$$

In general, $\bar{\Phi}(X_j, X_\ell, t)$ does not form a probabilistic distribution. So, we need to do the following normalization:

$$\hat{\Phi}(X_j, X_\ell, t) = \bar{\Phi}(X_j, X_\ell, t)/\bar{\bar{\Phi}}(X_j, t),$$

where

$$\bar{\bar{\Phi}}(X_j, t) = \sum \bar{\Phi}(X_j, X_\ell, t).$$

Combining the transition probabilities of diffusion and jump part, we have the following transition probabilities for ξ:

$$p^{(JD)}(X_j, X_\ell | q, b, \lambda) = (1 - \lambda\tau - o(\tau)) p^{(D)}(X_j, X_\ell | q, b) + (\lambda\tau + o(\tau)) \hat{\Phi}(X_j, X_\ell).$$

It is easy to verify that the transition probabilities $p^{(JD)}$ satisfy the locally jump-diffusion consistency conditions [8](see pg. 238). So the Markov chain $(\xi_t^{h,\tau})$ weakly converges to the surplus process X_t as $h, \tau \to 0$. What remains now is to solve the optimal control problem for the Markov chain $(\xi_t^{h,\tau})$.

Step 3. The solution of dynamic programming equation.

The value function of the Markov chain can be expressed as:

$$V^{h,\tau}(x, n\tau) = \sup_{(q(n\tau,x), b(n\tau,x))} E\left[u\left(\xi_{N_\tau}^{h,\tau}\right) | \xi_{n\tau} = x\right].$$

Using the transition probabilities $p^{(JD)}$ to calculate the expectation in the above equation, we obtain the dynamic programming equation:

$$V^{h,\tau}(x, n\tau) = \sup_{(q,b)\in[0,1]\times[0,1]} \left\{ \sum_y p^{(JD)}(x, y | q, b, \lambda) V^{h,\tau}(y, n\tau + \tau) \right\}. \quad (22)$$

This dynamic programming equation can be solved by a simple backward iteration procedure.

Now, the numerical algorithm for our optimal control problem has been established. Following the above steps, the value function and the optimal investment-reinsurance policy can be derived.

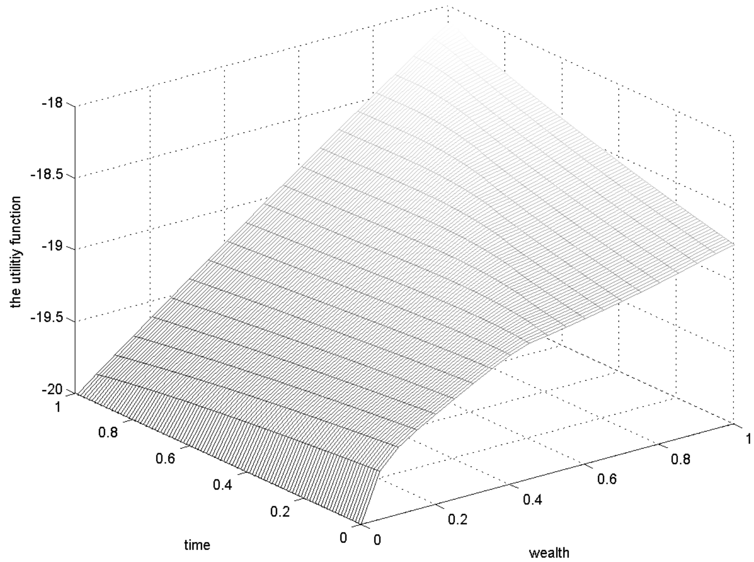

FIGURE 1. Value function as a function of time & wealth

Example. We will end this section with an example. The values of the parameters in this example are chosen just for illustrative purpose and no practical reasons other than that. Through this example, we show that the above numerical algorithm does work for the optimal investment-proportional reinsurance problem under no short-selling and no borrowing. We assume that the claim process of an insurance company is a compound Poisson process, in which the claim size has exponential distribution with parameter 1. As mentioned as in Section 1, the insurance company will purchase the proportional reinsurance from reinsurers and invest its surplus in the financial market.

We solve this optimal control problem for the insurance company using the numerical algorithm proposed above. With this numerical algorithm, the expected exponential utility of the terminal wealth as a function of time and initial wealth is depicted in the Figure 1. As in reality, the value function is increasing with respect to the wealth and it has the shape of a concave function, as in the case of diffusion models. The Figure 2 shows the relationship of the optimal investment fraction to the risky asset, time, and the initial wealth. For any fixed time t and when the surplus wealth is relatively small, the insurer is willing to take measures to earn more profit. In other words, the insurer is expected to invest in more risky asset when the wealth is small. In Figure 3 we can see the optimal reinsurance proportion as a function of time and the initial wealth. The optimal reinsurance proportion changes more dramatically. When the surplus wealth is small and the risk of investment is high, the insurer needs the reinsurance to reduce risk and prevent bankruptcy. In this case, the optimal reinsurance proportion is very high and almost reaches 1, i.e, the full reinsurance.

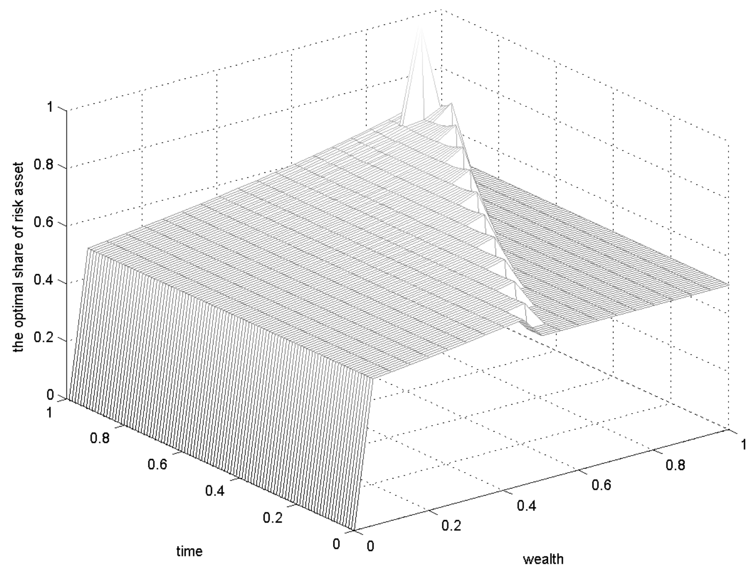

FIGURE 2. Optimal investment fraction as a function of time & wealth

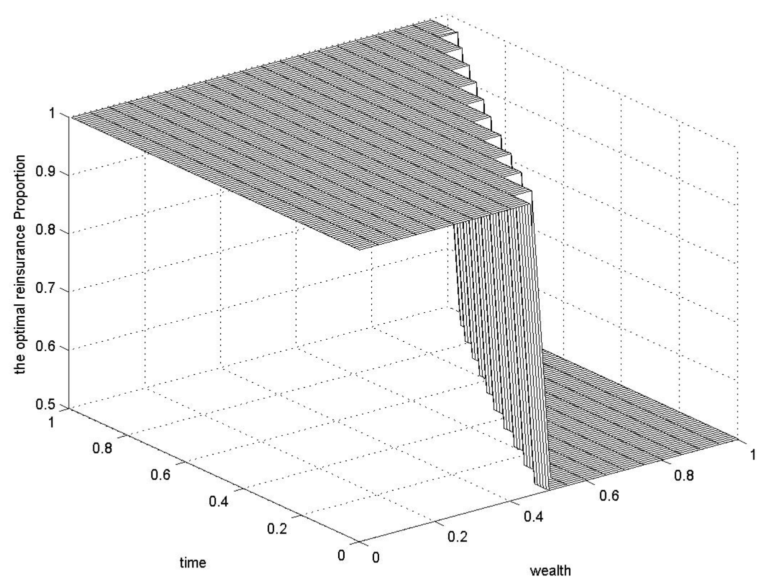

FIGURE 3. Optimal reinsurance proportion as a function of time & wealth

5. CONCLUDING REMARKS

In this paper, we study the optimal investment-proportional reinsurance problem disallowing short-selling and borrowing; this makes the model more realistic. There are other important factors that need to be considered in order to consider an even more realistic model, for example, the transaction costs for investment, illiquidity in the reinsurance markets, and the dividend payment. The more realistic the model is, less solvable the model becomes. This is the research direction of our future work.

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