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NON-AUTONOMOUS DIFFERENCE EQUATIONS: GLOBAL ATTRACTOR IN A BUSINESS-CYCLE MODEL WITH ENDOGENOUS POPULATION GROWTH

DAVID NIKOLAI CHEBAN, CRISTIANA MAMMANA, AND ELISABETTA MICHETTI

Department of Mathematics and Informatics, State University of Moldova,
Chişinău, Moldova
Department of Economics and Law, University of Macerata, Macerata, Italy
Department of Economics and Law, University of Macerata, Macerata, Italy

ABSTRACT. The article is devoted to the study of global attractors of quasi-linear non-autonomous difference equations and their structure. The results obtained are applied to the study of a two-dimensional triangular economic growth model of Solow type with Variable Elasticity of Substitution production function and endogenous population growth rate described by the Beverton-Holt equation.

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1. INTRODUCTION

The present paper is dedicated to the study of global attractors of quasi-linear non-autonomous difference equations

$$u_{n+1} = A(\sigma^n \omega)u_n + F(u_n, \sigma^n \omega), \quad (A \in C(\Omega, [E]),\ F \in C(E \times \Omega, E))$$

where Ω is a metric space, E is a finite-dimensional Banach space, $(\Omega, \mathbb{Z}_+, \sigma)$ is a dynamical system with discrete time $\mathbb{Z}_+$, $[E]$ is the space of all the linear operators acting on E equipped with operator norm, $C(\Omega, [E])$ (respectively, $C(E \times \Omega, E)$) is the space of all the continuous functions defined on Ω (respectively, on $E \times \Omega$) with values in $[E]$ (respectively, E) equipped with compact-open topology and F is a "small" perturbation. An analogous problem it was studied by Cheban D. et al. [11], when Ω is an invariant set. In this work a more general case is considered, when Ω is not invariant, but there exists a compact invariant subset $J \subseteq \Omega$ (Levinson center) attracting every compact subset from Ω.

The results obtained are applied to the study of a special class of triangular maps describing a discrete-time growth model of the Solow type where workers and shareholders have different but constant saving rates as in Bohm and Kaas [7] and the population growth rate evolution is described by the Beverton-Holt (BH) equation (see [1]).

Differently from previous works, we consider a Variable Elasticity of Substitution (VES) production function as proposed by Brianzoni et al. in [5], while assuming that the population growth rate evolves according to the BH equation as investigated in Brianzoni et al. [2] and Cheban et al. [11].

Our main goal is to study the long run dynamics of the economic model to show that, for suitable values of the parameters, it admits a compact global attractor and to describe its structure.

This paper is organized as follows.

In Section 2 we collect some notions and facts from the theory of dynamical systems (semigroup dynamical system, cocycle, full trajectory, non-autonomous dynamical system, compact global attractor) used in our paper. We also give some results of the existence of compact global attractors of quasi-linear dynamical systems.

Section 3 is dedicated to the study of a special class of the triangular maps $T : \mathbb{R}^2_+ \to \mathbb{R}^2_+$ describing an economic growth model with endogenous population growth rate. The analysis is performed applying the general results previously reached.

Section 4 concludes our paper.

2. SOME NOTIONS AND FACTS FROM DYNAMICAL SYSTEMS

In this Section we put together some notions and facts from the theory of dynamical systems (both with continuous and discrete time) that are used in our paper.

Let W and Ω be two complete metric spaces and denote by $X := W \times \Omega$ their Cartesian product. Recall [9, 20] that a continuous map $F : X \to X$ is called triangular if there are two continuous maps $f : W \times \Omega \to W$ and $g : \Omega \to \Omega$ such that $F = (f, g)$, i.e., $F(x) = F(u, \omega) = (f(u, \omega), g(\omega))$ for all $x =: (u, \omega) \in X$.

Consider a system of difference equations

$$\begin{cases} u_{n+1} = f(u_n, \omega_n) \\ \omega_{n+1} = g(\omega_n), \end{cases} \tag{2.1}$$

for all $n \in \mathbb{Z}_+$, where $\mathbb{Z}$ is the set of all integer numbers and $\mathbb{Z}_+ := \{n \in \mathbb{Z} : \ n \geq 0\}$.

Along with system (2.1) we consider the family of equations

$$u_{n+1} = f(u_n, g^n\omega) \ (\omega \in \Omega), \tag{2.2}$$

which is equivalent to system (2.1). Let $\varphi(n, u, \omega)$ be a solution of equation (2.2) passing through the point $u \in W$ for $n = 0$. It is easy to verify that the map $\varphi : \mathbb{Z}_+ \times W \times \Omega \to W$ $((n, u, \omega) \mapsto \varphi(n, u, \omega)$) satisfies the following conditions:

1. $\varphi(0, u, \omega) = u$ for all $u \in W$ and $\omega \in \Omega$;

2. $\varphi(n+m,u,\omega)=\varphi(n,\varphi(m,u,\omega),\sigma(m,\omega))$ for all $n,m\in\mathbb{Z}_+$, $u\in W$ and $\omega\in\Omega$, where $\sigma(n,\omega):=g^n\omega$;
3. the map $\varphi:\mathbb{Z}_+\times W\times\Omega\to W$ is continuous.

Denote by (Ω,Z_+,σ) the semigroup dynamical system generated by positive powers of the map $g:\Omega\to\Omega$, i.e., $\sigma(n,\omega):=g^n\omega$ for all $n\in\mathbb{Z}_+$ and $\omega\in\Omega$.

Recall [8, 19] that a triple $\langle W,\varphi,(\Omega,Z_+,\sigma)\rangle$ (in brief φ) is called a cocycle over the semigroup dynamical system $(\Omega,\mathbb{Z}_+,\sigma)$ with fiber W.

Let $X:=W\times\Omega$ and $(X,\mathbb{Z}_+,\pi)$ be a semigroup dynamical system on X, where $\pi(n,(u,\omega)):=(\varphi(n,u,\omega),\sigma(n,\omega))$ for all $u\in W$ and $\omega\in\Omega$, then $(X,\mathbb{Z}_+,\pi)$ is called [19] a skew-product dynamical system, generated by the cocycle $\langle W,\ \varphi,\ (\Omega,\mathbb{Z}_+,\ \sigma)\rangle$.

Remark 2.1. Then, the reasoning above shows that every triangular map generates a cocycle and, obviously, vice versa, i.e., if we have a cocycle $\langle W,\varphi,(\Omega,\mathbb{Z}_+,\sigma)\rangle$, we can define a triangular map $F:W\times\Omega\to W\times\Omega$ by the equality

$$F(u,\omega):=(f(u,\omega),g(\omega)),$$

where $f(u,\omega):=\varphi(1,u,\omega)$ and $g(\omega):=\sigma(1,\omega)$ for all $u\in W$ and $\omega\in\Omega$. The semigroup dynamical system defined by the positive powers of the map $F:X\to X$ $(X:=W\times\Omega)$ coincides with the skew-product dynamical system, generated by the cocycle $\langle W,\varphi,(\Omega,\mathbb{Z}_+,\sigma)\rangle$.

Taking into consideration this remark we can study triangular maps in the framework of cocycles with discrete time.

Let $(X,\mathbb{Z}_+,\pi)$ (respectively, $\langle W,\varphi,(\Omega,\mathbb{Z}_+,\sigma)\rangle$) be a semigroup dynamical system (respectively, a cocycle). A map $\gamma:\mathbb{Z}\to X$ is called an entire trajectory of the semigroup dynamical system $(X,\mathbb{Z}_+,\sigma)$ passing through the point $x\in X$ if $\gamma(0)=x$ and $\gamma(n+m)=\pi(m,\gamma(n))$ for all $n\in\mathbb{Z}$ and $m\in\mathbb{Z}_+$.

Let Ω be a complete metric space, $(X,\mathbb{Z}_+,\pi)$ (respectively, $(\Omega,\mathbb{Z}_+,\sigma)$) a semigroup dynamical system on X (respectively, Ω), and $h:X\to\Omega$ a homomorphism of $(X,\mathbb{Z}_+,\pi)$ onto $(\Omega,\mathbb{Z}_+,\sigma)$. Then the triple $\langle(X,\mathbb{Z}_+,\pi),\ (\Omega,\mathbb{Z}_+,\sigma),h\rangle$ is called a non-autonomous dynamical system (NDS).

Let W and Ω be complete metric spaces, $(\Omega,\mathbb{Z}_+,\sigma)$ a semigroup dynamical system on Y and $\langle W,\varphi,(\Omega,\mathbb{Z}_+,\sigma)\rangle$ a cocycle over $(\Omega,\mathbb{Z}_+,\sigma)$ with fiber W (or, in short, φ). We denote by $X:=W\times\Omega$ and define on X a skew product dynamical system $(X,\mathbb{Z}_+,\pi)$ ($\pi=(\varphi,\sigma)$, i.e., $\pi(t,(w,\omega))=(\varphi(t,w,\omega),\sigma(t,\omega))$ for all $t\in\mathbb{Z}_+$ and $(w,\omega)\in W\times\Omega$). Then the triple $\langle(X,\mathbb{Z}_+,\pi),((\Omega,\mathbb{Z}_+,\sigma),h\rangle$ is a non-autonomous dynamical system generated by the cocycle φ), where $h=pr_2:X\mapsto\Omega$ is the projection on the second component.

Let $\mathfrak{M}$ be a family of subsets from X.

A semigroup dynamical system $(X, \mathbb{Z}_+, \pi)$ will be called $\mathfrak{M}$-dissipative if for every $\varepsilon > 0$ and $M \in \mathfrak{M}$ there exists $L(\varepsilon, M) > 0$ such that $\pi(n, M) \subseteq B(K, \varepsilon)$ for any $n \geq L(\varepsilon, M)$, where K is a certain fixed subset from X depending only on $\mathfrak{M}$. In this case we will call K an attracting set for $\mathfrak{M}$.

As far as the applications are concerned, the most important cases are those when K is bounded or compact and $\mathfrak{M} := \{\{x\} \mid x \in X\}$ or $\mathfrak{M} := C(X)$, or $\mathfrak{M} := \{B(x, \delta_x) \mid x \in X,\ \delta_x > 0\}$, or $\mathfrak{M} := B(X)$ where $C(X)$ (respectively, $B(X)$) is the family of all compact (respectively, bounded) subsets from X.

The system $(X, \mathbb{Z}_+, \pi)$ is called [8]:

– point dissipative if there exists $K \subseteq X$ such that for every $x \in X$

$$\lim_{n \to +\infty} \rho(\pi(n, x), K) = 0; \tag{2.3}$$

– compactly dissipative if the equality (2.3) takes place uniformly w.r.t. x on the compact subsets from X.

Let $(X, \mathbb{Z}_+, \pi)$ be a compactly dissipative semigroup dynamical system and K be an attracting set for $C(X)$. We denote by

$$J := \Omega(K) = \bigcap_{n \geq 0} \overline{\bigcup_{m \geq n} \pi(m, K)},$$

then the set J does not depend on the choice of K and is characterized by the properties of the semi-group dynamical system $(X, \mathbb{Z}_+, \pi)$. The set J is called a Levinson center of the semigroup dynamical system $(X, \mathbb{Z}_+, \pi)$.

Let $(X, \mathbb{Z}_+, \pi)$ be a dynamical system and $x \in X$. Denote by

$$\omega_x := \Omega(\{x\}) = \bigcap_{n \geq 0} \overline{\bigcup_{m \geq n} \pi(m, x)}$$

the ω-limit set of point x.

If $(X, \mathbb{Z}_+, \pi)$ is a two sided dynamical system (i.e., the map $\pi(1, \cdot) : X \mapsto X$ is a homeomorphism) then the set

$$\alpha_x = \bigcap_{n \leq 0} \overline{\bigcup_{m \leq n} \pi(m, x)}$$

is said to be α-limit set of x.

Let E be a finite-dimensional Banach space with norm $|\cdot|$ and $\langle E, \varphi, (\Omega, \mathbb{Z}_+, \sigma)\rangle$ be a cocycle over $(\Omega, \mathbb{Z}_+, \sigma)$ with fiber E (or shortly φ).

A cocycle φ is called:

- dissipative, if there exists a number $r > 0$ such that

$$\limsup_{t \to +\infty} |\varphi(t, u, \omega)| \leq r$$

for all $\omega \in \Omega$ and $u \in E$;

- uniformly dissipative, if there exists a number $r > 0$ such that

$$\limsup_{t\to+\infty} \sup_{\omega\in\Omega',|u|\le R} |\varphi(t,u,\omega)| \le r$$

for all compact subsets $\Omega^{'} \subseteq \Omega$ and $R > 0$.

Theorem 2.2 ([10]). *If the dynamical system $(\Omega, \mathbb{Z}_+, \sigma)$ is compactly dissipative and the cocycle φ is uniformly dissipative, then the skew-product dynamical system $(X, \mathbb{Z}_+, \pi)$ is compactly dissipative.*

Below we present some results on the existence of compact global attractors of quasi-linear dynamical systems.

Let $(\Omega, \mathbb{Z}_+, \sigma)$ be a semigroup dynamical system on Ω with discrete time.

Let W be a complete metric space. Denote by $C(\Omega, W)$ the space of all the continuous functions $f : \Omega \to W$ endowed with the compact-open topology, i.e., the uniform convergence on compact subsets in Ω.

Consider a linear equation

$$u_{n+1} = A(\sigma(n,\omega))u_n, \quad (\omega \in \Omega) \tag{2.4}$$

where $A \in C(\Omega, [E])$.

Let $U(n, \omega)$ be the Cauchy operator of linear equation (2.4).

We will say that equation (2.4) is uniformly exponential stable if there exist constants $0 < q < 1$ and $N > 0$ such that

$$\|U(n,\omega)\| \le Nq^n$$

for all $\omega \in \Omega$ and $n \in \mathbb{Z}_+$.

Consider a difference equation

$$u_{n+1} = \mathcal{F}(u_n, \sigma(n,\omega)) \quad (\omega \in \Omega). \tag{2.5}$$

Denote by $\varphi(n, u, \omega)$ a unique solution of equation (2.5) with the initial condition $\varphi(0, u, \omega) = u$.

Equation (2.5) is said to be dissipative (respectively, uniformly dissipative), if the cocycle φ generated by equation (2.5) is so, i.e., there exists a positive number r such that

$$\limsup_{n\to+\infty} |\varphi(n,u,\omega)| \le r \quad \text{(respectively, } \limsup_{n\to+\infty} \sup_{\omega\in\Omega',|u|\le R} |\varphi(n,u,\omega)| \le r)$$

for all $u \in E$ and $\omega \in \Omega$ (respectively, for all compact subset $\Omega' \subseteq \Omega$ and $R > 0$).

Consider a quasi-linear equation

$$u_{n+1} = A(\sigma(n,\omega))u_n + F(u_n, \sigma(n,\omega)), \tag{2.6}$$

where $A \in C(\Omega, [E])$ and the function $F \in C(E \times \Omega, E)$ satisfies "the condition of smallness" (condition (ii) in Theorem 2.3).

Denote by $U(n, \omega)$ the Cauchy matrix for the linear equation

$$u_{n+1} = A(\sigma(n, \omega))u_n.$$

Theorem 2.3 ([10])**.** *Suppose that the following conditions hold:*

1. *equation* (2.4) *is uniformly exponential stable, i.e., there are positive numbers* N *and* $q < 1$ *such that*

$$\|U(n, \omega)\| \leq Nq^n \qquad (n \in Z_+); \tag{2.7}$$

2. $|F(u, \omega)| \leq C + D|u|$ $(C \geq 0,\ 0 \leq D < (1 - q)N^{-1})$ *for all* $u \in E$ *and* $\omega \in \Omega$.

Then equation (2.6) *is uniformly dissipative.*

Theorem 2.4 ([9])**.** *Let* (Ω, Z_+, σ) *be a compactly dissipative system and* φ *be a cocycle generated by equation* (2.6)*. Under the conditions of Theorem* 2.3 *the skew-product system* (X, Z_+, π) *(*$X := E \times \Omega$ *and* $\pi := (\varphi, \sigma)$*), generated by the cocycle* φ *admits a compact global attractor.*

Remark 2.5. Theorems 2.3 and 2.4 remain true if we replace the phase space E by positively invariant (with respect to the cocycle φ generated by (2.5)) subset $V \subset E$.

3. A BUSINESS–CYCLE MODEL WITH VES TECHNOLOGY AND POPULATION GROWTH RATE

3.1. **The Model.** The Solow-Swan growth model (see [21] and [22]) with Variable Elasticity of Substitution (VES) technology was firstly studied by Karagiannis et al. [15] assuming continuous time. The authors showed that the model can exhibit unbounded endogenous growth despite the absence of exogenous technical change and the presence of non-reproducible factors. However, their model is unable to produce economic fluctuations.

More recently, in [16], Palivos and Karagiannis proved that the variable elasticity of substitution plays a key role in the growth process. Following this contribution, in [5], Brianzoni et al. studied the discrete time Solow-Swan growth model, where the two types of agents, workers and shareholders, have different but constant saving rates as in Bohm and Kaas [7] and where the production function $f : R_+ \to R_+$, mapping capital per worker u into output per worker $f(u)$, is of the VES type. More precisely, following [16], they considered the specification of the VES production function in intensive form given by Revamkar [17] as follows:

$$f(u) = Au^{a\gamma}[1 + bau]^{(1-a)\gamma} \ (u \geq 0), \tag{3.1}$$

being $A > 0$, $a \in (0,1]$, $b \geq -1$ and $1/u \geq -b$, while assuming that the production function exhibits constant return to scale, i.e., $\gamma = 1$ and that the labor force grows at a constant rate.

The hypothesis of constant population growth rate is usually assumed in standard economic growth theory, however, this assumption is unable to explain possible fluctuations in the growth rate. For this reason a number of economic growth model with endogenous population growth have been proposed (see, for instance, Brianzoni et al. [2, 3, 4]). In particular Brianzoni et al. [2] and Cheban et al. [11] investigated the neoclassical growth model with differential saving and Constant Elasticity of Substitution (CES) production function under the assumption that the labor force dynamics is described by the BH equation (see Beverton and Holt [1]). The relevance of this iteration scheme is due to the fact that the BH equation is the solution in discrete time of the logistic model that describes a density dependent population growth mechanism having the following realistic economic properties: 1) when population is small in proportion to environmental carrying capacity, then it grows at a positive constant rate, 2) when population is larger in proportion to environmental carrying capacity, the resources become relatively more scarce and, as a result, the population growth rate is negatively affected.

In the present work we consider the Solow-Swan growth model in discrete time with differential saving and VES production function as proposed by Brianzoni et al. in [5] while assuming that the population growth rate evolves according to the BH equation as in Brianzoni et al. [2] and Cheban et al. [11]. Our main goal is to study the long run dynamics of the economic model in order to show that a compact global attractor is owned and to describe its structure.

Let us consider the following equation describing the evolution of the capital per capita u in the standard neoclassical Solow-Swan growth model with differential saving (see [7]):

$$F(u,\omega) = \frac{1}{1+\omega}[(1-\delta)u + s_w w(u) + s_r u f'(u)], \tag{3.2}$$

where $\delta \in (0,1)$ is the depreciation rate of capital, $s_w \in (0,1)$ and $s_r \in (0,1)$ are the constant saving rates for workers and shareholders respectively. The wage rate equals the marginal product of labor which is $w(u) := f(u) - uf'(u)$, furthermore shareholders receive the marginal product of capital $f'(u)$, which implies that the total capital income per worker is $uf'(u)$.

Observe that $\omega \geq 0$ represents the labor force growth rate: in our formulation we let it vary with time. More precisely we add a further assumption, that is the population growth rate evolves according to the BH equation given by:

$$\omega' = \frac{rh\omega}{h + (r-1)\omega}$$

where $h > 0$ is the carrying capacity (for example resource availability) and $r > 0$ is the inherent growth rate (this rate being determined by life cycle and demographic properties such as birth rate etc.).

By substituting the VES production function given by (3.1) (with $\gamma = 1$) in (3.2) we obtain the following map describing the evolution of capital accumulation:

$$H(u,\omega) = \frac{1}{1+\omega}\{(1-\delta)u + Au^a(1+abu)^{-a}[s_w(1-a) + s_r(a+abu)]\} \tag{3.3}$$

The resulting system, $T = (\omega', u')$, describing capital per worker (u) and population growth rate (ω) dynamics, is given by:

$$T := \begin{cases} u' = \frac{1}{1+\omega}\left[(1-\delta)u + Au^a(1+abu)^{-a}[s_w(1-a) + s_r(a+abu)]\right] \\ \\ \omega' = \frac{rh\omega}{h+(r-1)\omega}. \end{cases} \tag{3.4}$$

We get a discrete-time dynamical system described by the iteration of a map of the plane of triangular type. In fact, the second component of the previous system does not depend on u, therefore the map is characterized by the triangular structure:

$$T := \begin{cases} u' = g(u,\omega) \\ \omega' = f(\omega) \end{cases}. \tag{3.5}$$

As a consequence, the dynamics of the map T are influenced by the dynamics of the one–dimensional map f, that is the Beverton-Holt map.

3.2. **Dynamics of the Beverton-Holt Map** $f(\omega) = \frac{rh\omega}{h+(r-1)\omega}$**.** In this Subsection we study the dynamics of the one-dimensional monotone dissipative dynamical systems $(\mathbb{R}_+, \mathbb{Z}_+, \pi)$, generated by a strictly monotone increasing map $f : \mathbb{R}_+ \mapsto \mathbb{R}_+$.

Consider a continuous mapping $f : \mathbb{R}_+ \mapsto \mathbb{R}_+$.

Theorem 3.1. *Suppose that the following conditions are fulfilled:*

1. $f(0) = 0$*;*
2. f *is strictly monotone increasing;*
3. *the function* f *is bounded on* $\mathbb{R}_+$*;*
4. *there exists a number* $\alpha > 0$ *such that* $f(\alpha) > \alpha$*.*

Then the following statements hold:

1. *there exists a number* $x_0 > \alpha$ *such that* $f(x_0) = x_0$*;*
2. *the dynamical system* $(\mathbb{R}_+, f)$ *is point dissipative and* $\omega_x \subseteq [0, b]$ *for all* $x \in \mathbb{R}_+$*, where* $b = \lim_{x\to\infty} f(x)$*;*
3. *the dynamical system* $(\mathbb{R}_+, f)$ *admits a compact global attractor* $J \subset \mathbb{R}_+$*;*
4. $J = [0, x_0]$*, where* x_0 *is some fixed point of* f*;*
5. $\omega_x = \{x_0\}$ *for all* $x > x_0$*;*

6. *for any* $x \in (0, x_0)$ *there exist two fixed points* p *and* q *of the map* f *such that* $\lim\limits_{n\to\infty} f^n(x) = p$ *and* $\lim\limits_{n\to\infty} f^{-n}(x) = q$*;*
7. *if the mapping* f*, in addition, is strictly convex (i.e., the set* $G_f := \{(x, y) : x \in \mathbb{R}_+ \text{ and } 0 \le y \le f(x)\}$ *is strictly convex in* $\mathbb{R}^2$*), then*
   (a) x_0 *is a unique positive fixed point of the mapping* f*;*
   (b) $\lim\limits_{n\to\infty} f^{-n}(x) = 0$ *for all* $x \in [0, x_0)$*;*
   (c) $\lim\limits_{n\to\infty} f^n(x) = x_0$ *for all* $x > 0$*;*
   (d) *the fixed point* x_0 *is Lyapunov stable, i.e., for all* $\varepsilon > 0$ *there exists a number* $\delta = \delta(\varepsilon) > 0$ *such that* $|x - x_0| < \delta$ *implies* $|f^n(x) - x_0| < \varepsilon$ *or all* $n \ge 0$*;*
   (e) *the point* 0 *is Lyapunov stable in the negative direction, i.e., for all* $\varepsilon > 0$ *there exists a number* $\delta = \delta(\varepsilon) > 0$ *such that* $0 \le x < \delta$ *implies* $0 \le f^{-n}(x) < \varepsilon$ *or all* $n \ge 0$.

Proof. Consider the function $g(x) := f(x) - x$ and note that $g(\alpha) > 0$ and $g(\beta) < 0$ for all sufficiently large β ($\beta > b$) and, consequently, there exists $x_0 \in (\alpha, \beta)$ such that $g(x_0) = 0$ or $f(x_0) = x_0$.

Let $x \in \mathbb{R}_+$ be an arbitrary point. Since the semi-trajectory $\Sigma_x^+ := \{x, f(x), \ldots, f^n(x), \ldots\} \subseteq \{x\} \bigcup [0, b]$ is relatively compact, then the set ω_x is nonempty, compact and invariant. Let $q \in \omega_x$, then there exists a sequence $\{n_k\} \subset \mathbb{Z}_+$ such that $q = \lim\limits_{k\to+\infty} f^{n_k}(x)$ and $n_k \to \infty$ as $k \to \infty$ and, consequently, $q \in [0, b]$. Thus the dynamical system $(\mathbb{R}_+, f)$ is point dissipative.

Since the phase space $\mathbb{R}_+$ of the dynamical system $(\mathbb{R}_+, f)$ is local compact, then by Theorem 1.10 [8, ChI] it is compactly dissipative and by Theorem 1.6 [8, ChI] $(\mathbb{R}_+, f)$ admits a compact global attractor J (J is its maximal compact invariant set).

According to Theorem 1.32 [8, ChI] the global attractor J (Levinson center) of $(\mathbb{R}_+, f)$ is connected because the phase space $\mathbb{R}_+$ is so. On the other hand 0 is a fixed point and, consequently, $0 \in J$. Thus $0, \alpha \in J$ and, consequently, there exists a number $x_0 \ge \alpha > 0$ such that $J = [0, x_0]$. To complete this statement it is sufficient to show that $f(x_0) = x_0$. Note that the boundary $\partial J = \{0, x_0\}$ of the invariant set J is also invariant. In particular this means that $f(x_0) = x_0$ or $f(x_0) = 0$. Since the mapping f is strictly monotone decreasing and $x_0 > 0$, then the equality $f(x_0) = 0$ is not possible and, consequently, $f(x_0) = x_0$.

Consider $x > x_0$. Since the set $J = [0, x_0]$ is invariant, then $f^n(x) > x_0$ for all $n \in \mathbb{Z}_+$ and, consequently, $\omega_x \subset [x_0, +\infty)$. On the other hand $\omega_x \subseteq J = [0, x_0]$ and, consequently, $\omega_x \subseteq [x_0, +\infty) \bigcap [0, x_0] = \{x_0\}$.

Let $x \in (0, x_0)$ be an arbitrary number. Since the function f is strictly monotone increasing and bounded on $\mathbb{R}_+$, then there exists a limit $b := \lim\limits_{x\to\infty} f(x)$ and the reverse function $f^{-1} : [0, b) \mapsto \mathbb{R}_+$ is also strictly monotone increasing. Consider the sequence

$\{f^n(x)\}$. We will show that the sequence $\{f^n(x)\}$ is monotone. In fact, if $f(x) > x$ (respectively $f(x) < x$), then $f^{n+1}(x) > f^n(x)$ (respectively, $f^{n+1}(x) < f^n(x)$) for all $n \in \mathbb{N}$. Since J is invariant, then $f^n(x) \in J$ for all $n \in \mathbb{N}$. Thus the sequence $\{f^n(x)\}$ is bounded and monotone and, consequently, there exists $\lim\limits_{n\to\infty} f^n(x) = p$. It is easy to check that $f(p) = p$. Taking into account that the mapping f is an homeomorphism on the set J and it is invariant, by reasoning as above it is easy to show that the sequence $\{f^{-n}(x)\}$ is monotone and bounded and, consequently, there exists $\lim\limits_{n\to\infty} f^{-n}(x) = q$ and $f(q) = q$.

Suppose now that the function f is also strictly convex. We will show that in this case x_0 is a unique positive fixed point of f. Suppose that it is not so, then there exists a fixed point $\bar{x} \in (0, x_0)$. Note that the points $(0, 0), (\bar{x}, \bar{x})$ and (x_0, x_0) belong to G_f and $\Delta^+ := \{(x, x) :\ x \in \mathbb{R}_+\}$. Thus $(\bar{x}, \bar{x}) \in G_f \bigcap \Delta^+$ and $\bar{x} = \lambda x_0$, where λ is a number from $(0, 1)$. The last inclusion contradicts to the strictly convexity of the set G_f. The contradiction obtained proves our statement.

To complete the proof of the theorem it is sufficient to show that the point x_0 (respectively, point 0) is Lyapunov stable in the positive (respectively, negative) direction. Note that the set $A := \{x_0\}$ (respectively, $B := \{0\}$) is a locally maximal compact invariant set of the map f. Now Lyapunov stability in the positive direction (respectively, in the negative direction) of the point x_0 (respectively, 0) follows from Theorem 8.2 [8, ChVIII]. □

Remark 3.2. Note that the item (iv) of Theorem 3.1 remains true without the assumption that the mapping f is bounded. It is sufficient to suppose that the dynamical system $(f, \mathbb{R}_+)$ is compactly dissipative and f is strictly monotone increasing.

In what follows we assume $r > 1$ in order for an economic analysis to be of interests.

Lemma 3.3. *Consider* $f(x) := \frac{hrx}{h+(r-1)x}$ *for all* $x \in \mathbb{R}_+$, $h > 0$ *and* $r > 1$, *then the following statements hold:*

1. $f'(x) = \frac{rh^2}{(h+(r-1)x)^2}$ *for all* $x > 0$;
2. $f''(x) = \frac{-2r(r-1)h^2}{(h+(r-1)x)^3}$ *for all* $x > 0$;
3. $f(\alpha) > \alpha$, *where* $\alpha := h/2$.

Proof. This statement is evident. □

Corollary 3.4. *Consider* $f(x) := \frac{hrx}{h+(r-1)x}$ *for all* $x \in \mathbb{R}_+$, $h > 0$ *and* $r > 1$, *then the following statements hold:*

1. *the mapping* f *is strict monotone increasing and convex;*
2. f *admits two fixed points* $x = 0$ *and* $x = h$;

3. *the fixed point* 0 *is asymptotically stable in the negative direction and* $W^u(0) := \{x \in \mathbb{R}_+ : \lim_{n\to\infty} f^{-n}(x) = 0\} = [0,h)$*;*
4. *the fixed point* h *is asymptotically Lyapunov stable in the positive direction and* $W^s(0) := \{x \in \mathbb{R}_+ : \lim_{n\to\infty} f^{n}(x) = h\} = (0,+\infty)$*;*
5. *the dynamical system* $(\mathbb{R}_+, f)$ *is compactly dissipative and its Levinson center (compact global attractor)* $J = [0,h]$*.*

Proof. This statement follows from Theorem 3.1 and Lemma 3.3. □

The mapping $f : \mathbb{R}_+ \mapsto \mathbb{R}_+$ is said to admit a holomorphic extension if there exists $\delta > 0$ and a holomorphic function $\tilde{f} : B_\delta \mapsto \mathbb{C}$ such that $\tilde{f}\big|_{\mathbb{R}_+} = f$, where $B_\delta := \bigcup_{r\geq 0}\{(x,y) : (x-r)^2 + y^2 < \delta^2\}$.

Theorem 3.5. *Under the conditions of Theorem* 3.1*, if the function* f *admits an olomorphic extension, then* f *has a finite number of fixed points.*

Proof. Consider the holomorphic function $F(z) := \tilde{f}(z) - z$ defined on B_δ. Denote by $Fix(f) := \{x \in \mathbb{R}_+ : f(x) = x\}$ and note that $Fix(f) \subset J = [0, x_0]$. On the other hand every point $z \in Fix(f)$ is a null of the holomorphic function F. Since holomorphic function admits at most a finite number of nulls on every compact subset, then the set $Fix(f)$ contains at most a finite number of points. □

Thus, the dynamical system $(\mathbb{R}_+, f)$, generated by the Beverton-Holt map f admits a compact global attractor J for all $r > 1$. In addition J possesses the following property: if $h > 0$ and $r > 1$, then $J = [0,h]$ and in this case the fixed point $\omega = 0$ is a repeller (i.e., $\omega = 0$ is an asymptotically stable in the negative direction fixed point), but $\omega = h$ is an attractor with domain of attraction $(0,+\infty)$.

3.3. **Existence of a Compact Global Attractor.** In what follows we assume $b \in (0,+\infty)$.

Lemma 3.6. *The function* $H(u,\omega)$ *can be presented in the following form*

$$H(u,\omega) = \frac{1}{1+\omega}\{(1-\delta + s_r ab\frac{A}{(ab)^a})u\} + R(u,\omega), \tag{3.6}$$

where $R(u,\omega)$ *is bounded, i.e., there exists a positive constant* C *such that* $|R(u,\omega)| \leq C$ *for all* $\omega \in [0,+\infty)$ *and* $u \in [0,+\infty)$*.*

Proof. This statement can be proved with slight modification of the proof of Lemma 6.1 from [12]. □

Theorem 3.7. *Consider* $b > 0$ *and* $\delta > s_r ab\frac{A}{(ab)^a}$*, then the dynamical system* $(\mathbb{R}^2_+, T)$*, generated by map* (3.4)*, admits a compact global attractor* $J \subset \mathbb{R}^2_+$ *which possesses the following properties:*

1. *the set J is connected;*
2. *for every $\omega \in [0,h]$ the set $J_\omega := \{(x,\omega) : (x,\omega) \in J\}$ is connected;*
3. *for all $\omega \in (0,h)$ there exists a positive number b_ω such that $I_\omega = [0,b_\omega]$, where $I_\omega := pr_1(J_\omega)$;*
4. *for all $x := (u,\omega) \in \mathbb{R}^2_+$ we have:*
   (a) *$\omega_x \subseteq J_h$ if $\omega \neq 0$ (i.e., if $p \in \omega_x$, then $pr_2(p) = h$);*
   (b) *$\alpha_x \subseteq J_0$ if $x \in J$ and $\omega \neq h$ (this means that $pr_2(q) = 0$ for all $q \in \alpha_x$).*

Proof. Consider $b > 0$, then by Lemma 3.6 the function H can be written in the form

$$T_1(u,\omega) = \frac{1}{1+\omega}\{(1-\delta+s_r ab\frac{A}{(ab)^a})u\} + R(u,\omega), \tag{3.7}$$

where $R(u,\omega)$ is bounded, i.e., there exists a positive constant M such that $|R(u,\omega)| \leq M$ for all $(u,\omega) \in \mathbb{R}^2_+$.

Since $0 \leq \frac{1}{1+\omega} \leq 1$ for all $\omega \in \mathbb{R}_+$, then from (3.7) we obtain

$$0 \leq T_1(u,\omega) \leq \alpha u + M \tag{3.8}$$

for all $(u,\omega) \in \mathbb{R}^2_+$, where $\alpha := 1-\delta+s_r ab\frac{A}{(ab)^a} < 1$.

Since the map T is triangular, to prove the existence of compact global attractor J it is sufficient to apply Theorem 2.4 (see also Remark 2.5).

According to Theorem 1.32 from [8, ChI] the set J is connected. To prove the connectedness of the set J_ω we note that the map $f : \mathbb{R}_+ \mapsto \mathbb{R}_+$ ($f(\omega) := \frac{Hr\omega}{H+(r-1)\omega}$ for all $\omega \in \mathbb{R}_+$) is reversible, then by Theorem 2.25 [8, ChII] the set I_ω and, consequently, the set J_ω is also connected because $J_\omega = I_\omega \times \{\omega\}$. Thus for all $\omega \in [0,\omega]$ there are two numbers $a_\omega, b_\omega \in \mathbb{R}_+$ such that $I_\omega = [a_\omega, b_\omega]$. It easy to see that $a_\omega = 0$ for all $\omega \in [0,h]$ because $(0,\omega) \in J$ for all $\omega \in [0,H]$. At this point to complete the proof of this theorem it is sufficient to note that $b_\omega > 0$ for all $\omega \in (0,H)$. If we suppose that it is not so then there exists an $\bar{\omega} \in (0,H)$ such that $b_{\bar{\omega}} = 0$. From the last equality it follows that $\bar{\omega}$ is a fixed point of f (i.e., $f(\bar{\omega}) = \bar{\omega}$). The contradiction obtained proves our statement.

Let $x = (u,\omega) \in \mathbb{R}^2_+$ with the condition $\omega \neq 0$. Note that $\pi(t,x) = (\varphi(t,u,\omega), f^t\omega)$. Since $\omega \neq 0$, then $f^t\omega \to H$ as $t \to +\infty$. Let $p \in \omega_x$, then there exists a sequence $t_k \to +\infty$ ($t_k \in \mathbb{Z}_+$) such that $\pi(t_k,x) = (\varphi(t_k,u,\omega), f^{t_k}\omega) \to p$ as $k \to \infty$, i.e., $pr_2(p) = \lim_{k\to\infty} f^{t_k}\omega = H$. If $x \in J$ and $q \in \alpha_x$, then reasoning as above and taking into consideration that $\lim_{t\to\infty} f^{-t}\omega = 0$ (for all $\omega \in (0,H)$) we prove that $pr_2(q) = 0$. □

3.4. **Structure of the Attractor.** In this subsection we suppose that $b > 0$. Let us consider $H(u) = (1-\delta)u + f(u)[s_w(1-a) + s_r(a+abu)]$ and $H(u,\omega) = \frac{1}{1+\omega}H(u)$.

Lemma 3.8. *The following statements hold:*

1. *let* $f(u) := Au^a(1+abu)^{-a}$, *then*

$$f'(u) = \frac{af(u)}{u(1+abu)}; \tag{3.9}$$

2. *if* $H(u) = (1-\delta)u + f(u)[s_w(1-a) + s_r(a+abu)]$, *then*

$$H'(u) = 1-\delta + \Big(\frac{a}{u(1+abu)}[s_w(1-a)+s_r(a+abu)] + s_r ab\Big)f(u). \tag{3.10}$$

Proof. This statement is evident. □

Lemma 3.9. *The following statements hold:*

1. $H'(u) \geq 1-\delta > 0$ *for all* $u \in (0,+\infty)$;
2.

$$\lim_{u\to\infty} H'(u) = 1-\delta \quad \text{and} \quad \lim_{u\to 0^+} H'(u) = +\infty; \tag{3.11}$$

3. *there exists* $u_0 > 0$ *such that* $H(u) \geq (h+2)u$ *for all* $u \in [0, u_0]$.

Proof. The first and second statements are evident. To prove the third statement we note that from (3.11) it follows that, for given $h > 0$, there exists a positive number u_0 such that $H'(u) \geq h+2$ for all $u \in (0, u_0]$. Let now $\xi \in (0, u_0)$, then we have

$$H(u) - H(\xi) = H'(\theta)(u-\xi) \geq (h+2)(u-\xi) \tag{3.12}$$

for all $u \in (0, u_0)$, where $\theta \in (\xi, u)$. Passing into limit in (3.12) as $\xi \to 0$ and taking into account the continuity of $H(u)$ at the point $u = 0$ and the equality $H(0) = 0$, we obtain $H(u) \geq (h+2)u$ for all $u \in (0, u_0)$. Lemma is proved. □

Lemma 3.10. *Let* $(\mathbb{R}^2_+, T)$ *be a dynamical system generated by map* (3.4) *(i.e.,* $T^t(u,\omega) = (\varphi(t,u,\omega), f^t(\omega))$*) and* $\varphi(t,u,\omega) \in [0,u_0]$ *for all* $t \in \mathbb{Z}_+$, *then* $\varphi(t,u,\omega) \geq u$ *for all* $t \in \mathbb{Z}_+$, $u \in (0, u_0]$ *and* $\omega \in [0, h+1]$.

Proof. Note that $H(u,\omega) = \frac{1}{1+\omega}H(u)$ and $\varphi(t,u,\omega)$ is a unique solution of equation

$$u_{t+1} = H(u_t, f^t(\omega)) \tag{3.13}$$

with initial data $\varphi(0,u,\omega) = u$. Let $u \in [0,u_0]$, then by Lemma 3.9 we have $\varphi(1,u,\omega) = H(u,\omega) \geq u$ for all $u \in [0,u_0]$ because $\frac{1}{1+\omega} \in [\frac{1}{h+2}, 1]$ for all $\omega \in [0,h+1]$. Note that $f^t[0,h+1] \subseteq [0,h+1]$ for all $t \in \mathbb{Z}_+$. If we suppose that $\varphi(t,u,\omega) \geq u$ for all $t = 1,2,\ldots,n$, then we obtain $\varphi(t+1,u,\omega) = \varphi(1,\varphi(t,u,\omega), f^t(\omega)) \geq \varphi(t,u,\omega) \geq u$. Lemma is proved. □

Lemma 3.11 ([5]). *The following statements hold:*

1. *let* $\omega = \bar{\omega} \geq 0$, *if* $\frac{\omega+\delta}{A} > (ab)^{-a}abs_r$, *then* $H(\bar{\omega}, u)$ *has two fixed points:* $u_1 = 0$ *and* $u_2 = k^* > 0$*;*
2. *the fixed point* u_1 *(respectively,* u_2*) is locally unstable (respectively, stable).*

Let $(X, \mathbb{T}, \pi)$ be a dynamical system. The subset $A \subseteq X$ is said to be *chain transitive* (see [13, 14]) if, for any $a, b \in A$, and any $\varepsilon > 0$ and $L > 0$, there are finite sequences $x_1, x_2, \ldots, x_m \in A$ with $a = x_1, b = x_m$, and $t_1, t_2, \ldots, t_m \geq L$ such that $\rho(\pi(t_i, x_i), x_{i+1}) < \varepsilon$ $(1 \leq i \leq m-1)$. The sequence $\{x_1, x_2, \ldots, x_m\}$ is called an ε–chain in A connecting a and b.

Recall that the invariant set $M \subset X$ of dynamical system $(X, \mathbb{Z}_+, \pi)$ is said to be dynamically decomposable if there are two nonempty invariant subsets $M_i \subset M$ $(i = 1, 2)$ such that $M_1 \bigcap M_2 = \emptyset$ and $M = M_1 \bigcup M_2$. Otherwise the set M is said to be dynamically undecomposable.

Remark 3.12. 1. If the positive semi-trajectory $\Sigma_x^+ := \bigcup_{t \geq 0} \pi(t, x)$ is relatively compact, then its ω-limit set ω_x is chain transitive [6, 13, 18] (respectively, dynamically undecomposable [6]).

2. If the dynamical system $(X, \mathbb{Z}_+, \pi)$ is two-sided and the negative semi-trajectory $\Sigma_x^- := \bigcup_{t \leq 0} \pi(t, x)$ is relatively compact, then its α-limit set α_x is chain transitive [6, 13, 18] (respectively, dynamically undecomposable [6]).

Theorem 3.13. *Consider $\delta > s_r ab \frac{A}{(ab)^a}$, then the following statements hold:*

1. *the dynamical system $(f_0, \mathbb{Z}_+)$ (respectively, $(f_1, \mathbb{Z}_+)$) is compactly dissipative, where $f_0(u) := H(0, u)$ (respectively, $f_1(u) := H(h, u)$) for all $u \in \mathbb{R}_+$;*
2. *$J_0 = [0, k_0^*]$ (respectively, $J_1 = [0, k_1^*]$), where J_0 (respectively, J_1) is the Levinson center of the dynamical system $(f_0, \mathbb{Z}_+)$ (respectively, $(f_1, \mathbb{Z}_+)$) and k_0^* (respectively, k_1^*) is its positive fixed point;*
3. *for all $x = (\omega, u) \in \mathbb{R}_+^2$ with $\omega \neq 0, h$ and $u > 0$ we have $\omega_x = \{(h, k_1^*)\}$;*
4. *for all $x = (\omega, u) \in J$ with $\omega \in [0, h)$ we have $\alpha_x = \{(0, 0)\}$ or $\alpha_x = \{(0, k_0^*)\}$.*

Proof. The first statement follows from Theorem 3.7 because the set $\{(0, u) : u \in \mathbb{R}_+\}$ (respectively, $\{(h, u) : u \in \mathbb{R}_+\}$) is an invariant subset of the dynamical system $(\mathbb{R}_+^2, T)$.

Let J_0 (respectively, J_1) be the Levinson center of $(f_0, \mathbb{R}_+)$ (respectively, $(f_1, \mathbb{R}_+)$). Note the function f_0 (respectively, f_1) is strictly monotone increasing because

$$\partial_u H(\omega, u) > 0$$

for all $(\omega, u) \in \mathbb{R}_+^2$. The second statement of the theorem follows from Lemma 3.11, Theorem 3.1 (item (iv)) and Remark 3.2.

Let $x = (u, \omega) \in \mathbb{R}_+^2$ with $u > 0$ and $\omega \neq 0$ and $(\mathbb{R}_+^2, \pi)$ be a dynamical system generated by the triangular map T (see (3.5)), i.e., $\pi(t, x) = (\varphi(t, u, \omega), f^t \omega)$ for all $t \in \mathbb{Z}_+$ and $(u, \omega) \in \mathbb{R}_+^2$, where $\varphi(t, u, \omega)$ is a unique solution of equation

$$u(t+1) = H(f^t \omega, u(t))$$

passing through the point u at the initial moment $t = 0$. Since the function f is strictly monotone and $\partial_u H(u,\omega) > 0$, then the semigroup dynamical system $(\mathbb{R}^2_+, \pi)$ is two sided, i.e., every motion can be extended uniquely on $\mathbb{Z}$. Taking into account that the dynamical system $(\mathbb{R}^2_+, \pi)$ is compactly dissipative, then the positively semi-trajectory Σ^+_x is relatively compact, ω_x is a nonempty, compact, invariant and dynamically undecomposable set. Since the set ω_x is chain transitive, then $\omega_x = \{(k_1^*, h)\}$ or $\omega_x = \{(0,h)\}$. We will establish that the last equality is not possible. Suppose that $\omega_x = \{(0,h)\}$, then there exists a moment $t_0 \in \mathbb{Z}_+$ such that

$$f^t\omega \in [0, h+1] \quad \text{and} \quad \varphi(t,u,\omega) \in (0, u_0) \tag{3.14}$$

for all $t \geq t_0$. Taking into account (3.14) without loss of generality we can suppose that $t_0 = 0$ (if it is necessary we can take in the quality of $x = (u,\omega)$ the point $x_0 := \pi(t_0, x)$, because $\omega_x = \omega_{x_0}$). Since $\omega_x = \{(h,0)\}$, then we have

$$\lim_{t\to+\infty} \varphi(t,u,\omega) = 0. \tag{3.15}$$

On the other hand by Lemma 3.10 we have

$$\varphi(t,u,\omega) \geq u \tag{3.16}$$

for all $t \in \mathbb{Z}_+$. The conditions (3.15) and (3.16) are contradictory. The contradiction obtained proves our statement.

Let now $x = (u,\omega) \in J$ with $\omega \in [0,h)$, then by Theorem 3.7 we have $\alpha_x \subseteq J_0$. Note that the set α_x is chain transitive. On the other hand α_x is dynamically undecomposable and, consequently, $\alpha_x = \{(0,0)\}$ or $\alpha_x = \{(0, k_0^*)\}$. The theorem is completely proved. □

4. CONCLUSIONS

In the present paper we studied a discrete time neoclassical one-sector growth model with differential savings while assuming a VES production function and endogenous population growth rate described by the BH equation.

By proving some results on the existence of compact global attractors of quasi-linear dynamical systems, we showed that the growth model admits a compact global attractor if the elasticity of substitution between production factors is greater than one (i.e. $b > 0$) for suitable values of the parameters (for instance if the saving rate of shareholders s_r is low enough).

This result confirms the evidences reached in Brianzoni et al. [5] for the case of constant population growth rate. Furthermore, our setup can be compared with the model studied by Brianzoni et al. [2] and Cheban et al. [11] in which the Costant Elasticity of Substitution (CES) production function was into account: in both cases the economy has a compact global attractor where the asymptotic dynamics occurs.

Finally, differently from Cheban et al. [12] in which the evolution of the population growth rate was described by the logistic equation, once our model was considered in the interior of its domain (i.e. starting from initial conditions with both positive capital per capita and population growth rate), we found that it admits a unique globally asymptotically stable fixed point, providing that in the long run the economic growth will convergetoward a positive steady state, while fluctuations or more complex dynamics are ruled out.

The analysis proposed represents a first step in the study of the determinants of economic growth, when the elasticity of substitution between production factors and the population growth rate are not constant. A further step in this research line would take into account how long run dynamics change if population growth rate evolution depends also on the capital per capita level. The final discrete time dynamical system is no longer triangular, and a different analysis must be used to determine the qualitative and quantitative properties of economic growth.

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