POSITIVE SOLUTIONS FOR A CLASS OF SINGULAR THIRD-ORDER THREE-POINT NONHOMOGENEOUS BOUNDARY VALUE PROBLEM

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ABSTRACT. In this paper we establish the existence or nonexistence of positive solutions for singular third-order three-point nonhomogeneous boundary value problem. First, we give a new form of the solution, and then, some useful properties of the Green's function are obtained by a new method. Finally, we employ a cone theoretic fixed-point index theorem to establish our results.

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1. INTRODUCTION

Third-order differential equations arise in a variety of different areas of applied mathematics and physics, for example, in the deflection of a curved beam having a constant and varying cross section, a three-layer beam, electromagnetic waves or gravity driven flows and so on [6, 8, 15]. Recently, there is much attention being paid to the questions of positive solutions of third order three-point boundary value problem (BVP for short), see [1, 2, 3, 4, 5, 7, 9, 10, 13, 14, 16, 17] and the references therein.

Motivated greatly by the above-mentioned excellent works, here, we consider the existence and nonexistence of positive solutions for the following singular third order three-point nonhomogeneous BVP

$$u'''(t) + a(t)f(u(t)) = 0, \qquad 0 < t < 1, \tag{1.1}$$

$$u(0) = u''(0) = 0, \qquad u(1) - \alpha u(\eta) = \lambda, \tag{1.2}$$

where $0 < \eta < 1$, $0 < \alpha < 1/\eta$, $a(t)$ is allowed to be singular at $t = 0$ or $t = 1$, $\lambda \in (0, +\infty)$ is a parameter. In order to get positive solutions for BVP (1.1) and (1.2), it is assumed throughout that

(A_1) $a(t) \in C((0,1), [0, +\infty))$, $a(t)$ do not vanish identically on any subinterval of $(0,1)$, and $0 < \int_0^1 g(s)a(s)ds < +\infty$ ($g(s)$ will be given in Lemma 2.2);

(A_2) $f \in C([0,+\infty),[0,+\infty))$.

In this work we first give a new form of the solution, and then, some useful properties of the corresponding Green's function are obtained by a new method. Finally, by employing the fixed point index theorem and Schauder's fixed point theorem, some sufficient conditions guaranteeing the existence or nonexistence of positive solution if the nonlinearity f is either superlinear or sublinear are established to BVP (1.1) and (1.2).

The rest of the paper is organized as follows. In Section 2, we give some preliminaries and lemmas which are needed later. Then the main results on the existence or nonexistence of positive solution are presented in Section 3.

2. PRELIMINARIES

In this section, we present some preliminaries and lemmas which are useful to the proof of the main results.

Lemma 2.1. *For $y \in C[0,1]$, the BVP*

$$u'''(t) + y(t) = 0, \qquad 0 < t < 1, \tag{2.1}$$

$$u(0) = u''(0) = 0, \qquad u(1) - \alpha u(\eta) = \lambda \tag{2.2}$$

has a unique solution $u(t) = \int_0^1 G(t,s)y(s)ds + \frac{\alpha t}{2(1-\alpha\eta)}\int_0^1 G(\eta,s)y(s)ds + \frac{\lambda t}{1-\alpha\eta}$, *where*

$$G(t,s) = \frac{1}{2}\begin{cases}(1-t)(t-s^2), & 0 \le s \le t \le 1,\\ t(1-s)^2, & 0 \le t \le s \le 1.\end{cases} \tag{2.3}$$

Proof. In fact, if $u(t)$ is a solution of the BVP (2.1) and (2.2), then we may suppose that $u(t) = -\frac{1}{2}\int_0^t (t-s)^2 y(s)ds + At^2 + Bt + C$. By the boundary conditions (2.2), we have $A = C = 0$ and

$$B = \frac{1}{2(1-\alpha\eta)}\int_0^1 (1-s)^2 y(s)ds - \frac{\alpha}{2(1-\alpha\eta)}\int_0^\eta (\eta-s)^2 y(s)ds + \frac{\lambda}{1-\alpha\eta}.$$

As a result, BVP (2.1) and (2.2) has a unique solution

$$\begin{aligned}
u(t) &= -\frac{1}{2}\int_0^t (t-s)^2 y(s)ds + \frac{t}{2(1-\alpha\eta)}\int_0^1 (1-s)^2 y(s)ds \\
&\quad - \frac{\alpha t}{2(1-\alpha\eta)}\int_0^\eta (\eta-s)^2 y(s)ds + \frac{\lambda t}{1-\alpha\eta} \\
&= -\frac{1}{2}\int_0^t (t-s)^2 y(s)ds + \frac{t}{2}\int_0^1 (1-s)^2 y(s)ds + \frac{t\alpha\eta}{2(1-\alpha\eta)}\int_0^1 (1-s)^2 y(s)ds \\
&\quad - \frac{\alpha t}{2(1-\alpha\eta)}\int_0^\eta (\eta-s)^2 y(s)ds + \frac{\lambda t}{1-\alpha\eta} \\
&= \frac{1}{2}\int_0^t (1-t)(t-s^2)y(s)ds + \frac{1}{2}\int_t^1 t(1-s)^2 y(s)ds
\end{aligned}$$

$$+\frac{\alpha t}{2(1-\alpha\eta)}\int_0^\eta (1-\eta)(\eta-s^2)y(s)ds$$
$$+\frac{\alpha t}{2(1-\alpha\eta)}\int_\eta^1 \eta(1-s)^2y(s)ds+\frac{\lambda t}{1-\alpha\eta}$$
$$=\int_0^1 G(t,s)y(s)ds+\frac{\alpha t}{2(1-\alpha\eta)}\int_0^1 G(\eta,s)y(s)ds+\frac{\lambda t}{1-\alpha\eta}.$$

The proof is complete. □

Lemma 2.2. *For the function $G(t,s)$, we have the following results:*

(i) $0 \le G(t,s) \le g(s)$ *for* $(t,s) \in [0,1]\times[0,1]$, *where* $g(s)=\frac{(1-s^2)^2}{8}$;
(ii) $G(t,s) \ge \frac{1}{4}g(s)$ *for* $(t,s)\in[\frac{1}{4},\frac{3}{4}]\times[0,1]$.

Proof. (i) $G(t,s)\ge 0$ is obvious. Next we prove $G(t,s)\le g(s)$.

In fact, if $s\le t$, we find $G(t,s)=\frac{1}{2}(1-t)(t-s^2)=\frac{1}{2}\left(-t^2+(1+s^2)t-s^2\right)$ has a maximum value $\frac{(1-s^2)^2}{8}$, if $t\le s$, we know $G(t,s)=\frac{1}{2}t(1-s)^2\le(1-s)^2(1+s)^2/8=(1-s^2)^2/8$.

(ii) For any $(t,s)\in[\frac{1}{4},\frac{3}{4}]\times[0,1]$, if $s\le t$, from (2.3) we have

$$G(t,s)=\frac{1}{2}(1-t)(t-s^2)=\frac{t(1-s^2)^2}{8}+\frac{1}{2}\left[(1-t)(t-s^2)-\frac{t(1-s^2)^2}{4}\right],$$

since the quadratic function $x(t)=-t^2+(1+s^2-\frac{(1-s^2)^2}{4})t-s^2\ge 0$ for $t\in[1/4,3/4]$, so

$$G(t,s)=\frac{1}{2}(1-t)(t-s^2)\ge\frac{t(1-s^2)^2}{8}\ge\frac{1}{4}g(s).$$

If $t\le s$, then $G(t,s)=\frac{1}{2}t(1-s)^2\ge\frac{t(1-s)^2(1+s)^2}{8}\ge\frac{1}{4}g(s)$. The proof is complete. □

Let $E=C[0,1]$ with the norm $\|u\|=\max_{t\in[0,1]}|u(t)|$. Then, E is a Banach space. If we let

$$K=\left\{u\in E: u(t)\ge 0, t\in[0,1], \min_{1/4\le t\le 3/4}u(t)\ge\frac{1}{4}\|u\|\right\},$$

then K is a cone of E. Denote $K_r=\{u\in K \mid \|u\|<r\}$, $\partial K_r=\{u\in K\mid\|u\|=r\}$ for $r>0$.

For $u\in E$, we define the operator T by

(2.4)
$$Tu(t)=\int_0^1 G(t,s)a(s)f(u(s))ds+\frac{\alpha t}{2(1-\alpha\eta)}\int_0^1 G(\eta,s)a(s)f(u(s))ds+\frac{\lambda t}{1-\alpha\eta}.$$

By Lemma 2.1, BVP (1.1) and (1.2) has a solution $u=u(t)$ if and only if u is a fixed point of T.

Lemma 2.3 ([12])**.** *Suppose that E ia a Banach space, $T_n : E \to E$ $(n =, 1, 2, 3, \dots)$ are completely continuous operators, $T : E \to E$, and*

$$\lim_{n\to\infty} \max_{\|u\|<r} \|T_n u - Tu\| = 0 \quad \textit{for } r > 0,$$

then T is a completely continuous operator.

Lemma 2.4. *The operator T: $K \to K$ is completely continuous.*

Proof. It follows from (2.4) and Lemma 2.2, we know that for $u \in K$, $t \in [0, 1]$,

$$0 \le Tu(t) \le \int_0^1 g(s)a(s)f(u(s))ds + \frac{\alpha}{2(1-\alpha\eta)} \int_0^1 G(\eta, s)a(s)f(u(s))ds + \frac{\lambda}{1-\alpha\eta},$$

thus,

$$\|Tu\| \le \int_0^1 g(s)a(s)f(u(s))ds + \frac{\alpha}{2(1-\alpha\eta)} \int_0^1 G(\eta, s)a(s)f(u(s))ds + \frac{\lambda}{1-\alpha\eta}.$$

On the other hand, Lemma 2.2 imply that, for any $t \in [\frac{1}{4}, \frac{3}{4}]$,

$$\begin{aligned} Tu(t) &= \int_0^1 G(t, s)a(s)f(u(s))ds + \frac{\alpha t}{2(1-\alpha\eta)} \int_0^1 G(\eta, s)a(s)f(u(s))ds + \frac{\lambda t}{1-\alpha\eta} \\ &\ge \frac{1}{4}\left[\int_0^1 g(s)a(s)f(u(s))ds + \frac{\alpha}{2(1-\alpha\eta)} \int_0^1 G(\eta, s)a(s)f(u(s))ds + \frac{\lambda}{1-\alpha\eta}\right]. \end{aligned}$$

Therefore,

$$\min_{1/4\le t\le 3/4} Tu(t) \ge \frac{1}{4}\|Tu\|.$$

Hence, operator T satisfies $T(K) \subseteq K$.

In the following we prove that T is a completely continuous operator. For any natural number n $(n \ge 2)$, we define the function $a_n(t)$ by

$$a_n(t) = \begin{cases} \inf\{a(t), a(\frac{1}{n})\}, & 0 < t \le \frac{1}{n}, \\ a(t), & \frac{1}{n} \le t \le 1 - \frac{1}{n}, \\ \inf\{a(t), a(1-\frac{1}{n})\}, & 1 - \frac{1}{n} \le t < 1 \end{cases}$$

and operator $T_n : K \to K$ by

$$T_n u(t) = \int_0^1 G(t, s)a_n(s)f(u(s))ds + \frac{\alpha t}{2(1-\alpha\eta)} \int_0^1 G(\eta, s)a_n(s)f(u(s))ds + \frac{\lambda t}{1-\alpha\eta}.$$

Obviously, T_n is completely continuous on K for any $n \ge 2$ by an application of Ascoli-Arzela theorem. For a cone K_R, then T_n converges uniformly to T as $n \to \infty$. In fact, for any $t \in [0, 1]$, for each fixed $R > 0$ and $u \in K_R$, when $n \to \infty$, we get

$$\begin{aligned} &|T_n u(t) - Tu(t)| \\ &= \Big| \int_0^1 G(t, s)[a_n(s) - a(s)]f(u(s))ds \end{aligned}$$

$$+\frac{\alpha t}{2(1-\alpha\eta)}\int_0^1 G(\eta,s)[a_n(s)-a(s)]f(u(s))ds\Big|$$
$$\le \int_0^{1/n} g(s)|a_n(s)-a(s)|f(u(s))ds+\frac{\alpha t}{2(1-\alpha\eta)}\int_0^{1/n} g(s)|a_n(s)-a(s)|f(u(s))ds$$
$$+\int_{1-1/n}^1 g(s)|a_n(s)-a(s)|f(u(s))ds$$
$$+\frac{\alpha t}{2(1-\alpha\eta)}\int_{1-1/n}^1 g(s)|a_n(s)-a(s)|f(u(s))ds\to 0,$$

where we have used the fact that $G(t,s)\le g(s)$ for $0\le t,s\le 1$. Hence T_n converges uniformly to T as $n\to\infty$, and therefore T is completely continuous by Lemma 2.3. □

To prove our main results, we need the following lemma.

Lemma 2.5 ([11]). *Let $\varphi: K\to K$ be a completely continuous mapping and $\varphi u\ne u$ for $u\in\partial K_r$. Then we have the following conclusions:*

(1) *if $\|\varphi u\|\ge\|u\|$ for $u\in\partial K_r$, then $i(\varphi,K_r,K)=0$;*
(2) *if $\|\varphi u\|\le\|u\|$ for $u\in\partial K_r$, then $i(\varphi,K_r,K)=1$.*

3. MAIN RESULTS

In this section, we will state and prove our main results. Throughout this section, we shall use the following notation:

$$N_1=\left(\int_0^1 g(s)a(s)ds+\frac{\alpha}{2(1-\alpha\eta)}\int_0^1 g(s)a(s)ds\right)^{-1},$$
$$N_2=\left(\frac{1}{4}\int_{1/4}^{3/4} g(s)a(s)ds+\frac{\alpha}{8(1-\alpha\eta)}\int_{1/4}^{3/4} G(\eta,s)a(s)ds\right)^{-1}.$$

Then it is obvious that $0<N_1<N_2$. We define

$$f_0=\lim_{u\to 0^+}\frac{f(u)}{u},\qquad f_\infty=\lim_{u\to\infty}\frac{f(u)}{u}.$$

Theorem 3.1. *Assume that there is $r_1>0$ such that $f(u)<\frac{N_1}{2}r_1$ for $u\in[0,r_1]$. If $f_\infty=\infty$, then BVP (1.1) and (1.2) has at least one positive solution for λ small enough and has no positive solution for λ large enough.*

Proof. For r_1, let λ satisfy

$$0<\lambda<\frac{(1-\alpha\eta)r_1}{2}. \tag{3.1}$$

Then for any $u \in \partial K_{r_1}$, it follows from Lemma 2.2, (2.4), (3.1) and the assumption $f(u) < \frac{N_1}{2} r_1$ for $u \in [0, r_1]$ that for $t \in [0,1]$,

$$\begin{aligned} Tu(t) &= \int_0^1 G(t,s)a(s)f(u(s))ds + \frac{\alpha t}{2(1-\alpha\eta)} \int_0^1 G(\eta,s)a(s)f(u(s))ds + \frac{\lambda t}{1-\alpha\eta} \\ &\leq \int_0^1 g(s)a(s)f(u(s))ds + \frac{\alpha}{2(1-\alpha\eta)} \int_0^1 g(s)a(s)f(u(s))ds + \frac{\lambda}{1-\alpha\eta} \\ &< \frac{N_1 r_1}{2} \left(\int_0^1 g(s)a(s)ds + \frac{\alpha}{2(1-\alpha\eta)} \int_0^1 g(s)a(s)ds \right) + \frac{r_1}{2} \\ &= \frac{r_1}{2} + \frac{r_1}{2} = \|u\|, \end{aligned}$$

that is

$$\|Tu\| < \|u\| \quad \text{for } u \in \partial K_{r_1}. \tag{3.2}$$

On the other hand, since $f_\infty = \infty$, for N_2, there exists $R_1 > r_1$ such that

$$f(u) \geq 4N_2 u, \quad u \in [R_1/4, \infty). \tag{3.3}$$

Then for $u \in \partial K_{R_1}$, then $u(t) \geq \frac{\|u\|}{4} = \frac{R_1}{4}$, $t \in [\frac{1}{4}, \frac{3}{4}]$, so in view of Lemma 2.2, (2.4) and (3.3), we conclude that for $t \in [\frac{1}{4}, \frac{3}{4}]$,

$$\begin{aligned} Tu(t) &= \int_0^1 G(t,s)a(s)f(u(s))ds + \frac{\alpha t}{2(1-\alpha\eta)} \int_0^1 G(\eta,s)a(s)f(u(s))ds + \frac{\lambda t}{1-\alpha\eta} \\ &> \frac{1}{4} \int_{1/4}^{3/4} g(s)a(s)f(u(s))ds + \frac{\alpha}{8(1-\alpha\eta)} \int_{1/4}^{3/4} G(\eta,s)a(s)f(u(s))ds \\ &> \frac{1}{4} \int_{1/4}^{3/4} g(s)a(s)4N_2 u(s)ds + \frac{\alpha}{8(1-\alpha\eta)} \int_{1/4}^{3/4} G(\eta,s)a(s)4N_2 u(s)ds \\ &\geq N_2 \left(\int_{1/4}^{3/4} g(s)a(s)ds + \frac{\alpha}{2(1-\alpha\eta)} \int_{1/4}^{3/4} G(\eta,s)a(s)ds \right) \frac{\|u\|}{4} \\ &= \|u\|, \end{aligned}$$

which implies that

$$\|Tu\| > \|u\| \quad \text{for } u \in \partial K_{R_1}. \tag{3.4}$$

Therefore, by Lemma 2.5, (3.2), (3.4) and the property of fixed-point index, we obtain

$$i(T, K_{R_1} \backslash \overline{K_{r_1}}, K) = i(T, K_{R_1}, K) - i(T, K_{r_1}, K) = 0 - 1 = -1.$$

Thus the operator T has at least one fixed point $u \in K_{R_1} \backslash \bar{K}_{r_1}$, which is a positive solution of BVP (1.1) and (1.2).

Next we prove that BVP (1.1) and (1.2) has no positive solution for λ large enough. Otherwise, there exist $0 < \lambda_1 < \lambda_2 < \cdots < \lambda_n < \cdots$, with $\lim_{n\to\infty} \lambda_n = +\infty$, for any positive integer n, the problem

$$u'''(t) + a(t)f(u(t)) = 0, \qquad 0 < t < 1, \tag{3.5}$$

$$u(0) = u''(0) = 0, \qquad u(1) - \alpha u(\eta) = \lambda_n \tag{3.6}$$

has a positive solution u_n. By (2.4), we get

$$\begin{aligned} u_n(1) &= \int_0^1 G(1,s)a(s)f(u_n(s))ds + \frac{\alpha}{2(1-\alpha\eta)}\int_0^1 G(\eta,s)a(s)f(u_n(s))ds + \frac{\lambda_n}{1-\alpha\eta} \\ &\geq \frac{\lambda_n}{1-\alpha\eta} \to \infty, \quad (n \to \infty). \end{aligned}$$

Therefore, $\|u_n\| \to \infty$ as $n \to \infty$.

Since $f_\infty = \infty$, for N_2, there exists $R > 0$ such that

$$f(u) \geq 8N_2 u \quad \text{for } u \in [R/4, \infty).$$

Let n be large enough such that $\|u_n\| \geq R$. Then for $t \in [\frac{1}{4}, \frac{3}{4}]$, $u_n(t) \geq \frac{R}{4}$ and

$$\begin{aligned} \|u_n\| &\geq u_n(t) \\ &= \int_0^1 G(t,s)a(s)f(u_n(s))ds + \frac{\alpha t}{2(1-\alpha\eta)}\int_0^1 G(\eta,s)a(s)f(u_n(s))ds + \frac{\lambda_n t}{1-\alpha\eta} \\ &> \frac{1}{4}\left[\int_{1/4}^{3/4} g(s)a(s)f(u_n(s))ds + \frac{\alpha}{2(1-\alpha\eta)}\int_{1/4}^{3/4} G(\eta,s)a(s)f(u_n(s))ds\right] \\ &> \frac{1}{4}\left[\int_{1/4}^{3/4} g(s)a(s)8N_2u_n(s)ds + \frac{\alpha}{2(1-\alpha\eta)}\int_{1/4}^{3/4} G(\eta,s)a(s)8N_2u_n(s)ds\right] \\ &\geq 2N_2\left[\int_{1/4}^{3/4} g(s)a(s)ds + \frac{\alpha}{2(1-\alpha\eta)}\int_{1/4}^{3/4} G(\eta,s)a(s)ds\right]\frac{\|u_n\|}{4} \\ &= 2\|u_n\|, \end{aligned}$$

which is a contradiction. The proof is complete. □

Corollary 3.2. *If $f_0 = 0$, $f_\infty = \infty$, then BVP* (1.1) *and* (1.2) *has at least one positive solution for λ small enough and has no positive solution for λ large enough.*

Proof. The conclusion readily follows from Theorem 3.1. □

Theorem 3.3. *Suppose that the hypothesis of Theorem* 3.1 *hold. In addition, if f is nondecreasing, then there is $\Lambda > 0$ such that BVP* (1.1) *and* (1.2) *has at least one positive solution for $\lambda \in (0, \Lambda)$ and has no positive solution for $\lambda \in (\Lambda, \infty)$.*

Proof. Set $F = \{\lambda \mid \text{BVP (1.1) and (1.2) has at least one positive solution}\}$, let $\Lambda = \sup F$, it follows from Theorem 3.1 that $0 < \Lambda < \infty$. From the definition of Λ, we know for any $\lambda \in (0, \Lambda)$, there is a $\lambda^* > \lambda$ such that BVP

$$\begin{aligned} &u'''(t) + a(t)f(u(t)) = 0, \qquad 0 < t < 1, \\ &u(0) = u''(0) = 0, \qquad u(1) - \alpha u(\eta) = \lambda^* \end{aligned}$$

has a positive solution $u^*(t)$. Next we will prove that for any $\lambda \in (0, \lambda^*)$, BVP (1.1) and (1.2) has a positive solution.

In fact, let $P_{u^*} = \{u \in K \mid u(t) \leq u^*(t), t \in [0,1]\}$, for any $\lambda \in (0, \lambda^*)$, $u \in P_{u^*}$, it follows from (2.4) and the monotonicity of f we get

$$\begin{aligned} Tu(t) &= \int_0^1 G(t,s)a(s)f(u(s))ds + \frac{\alpha t}{2(1-\alpha\eta)} \int_0^1 G(\eta,s)a(s)f(u(s))ds + \frac{\lambda t}{1-\alpha\eta} \\ &\leq \int_0^1 G(t,s)a(s)f(u^*(s))ds + \frac{\alpha t}{2(1-\alpha\eta)} \int_0^1 G(\eta,s)a(s)f(u^*(s))ds + \frac{\lambda^* t}{1-\alpha\eta} \\ &= u^*(t). \end{aligned}$$

So $T(P_{u^*}) \subseteq P_{u^*}$, by Schauder's fixed point theorem we know that T has a fixed point $u \in P_{u^*}$, which is a positive solution of BVP (1.1) and (1.2). The proof is complete. □

Theorem 3.4. *Assume that $f_0 = \infty$, $f_\infty = 0$, then BVP (1.1) and (1.2) has at least one positive solution for $\lambda \in (0, \infty)$.*

Proof. Since $f_0 = \infty$, there exists $r_2 > 0$ such that $f(u) \geq 4N_2 u$ for $u \in [0, r_2]$. Let $u \in K_{r_2}$, then for $t \in [\frac{1}{4}, \frac{3}{4}]$,

$$\begin{aligned} Tu(t) &= \int_0^1 G(t,s)a(s)f(u(s))ds + \frac{\alpha t}{2(1-\alpha\eta)} \int_0^1 G(\eta,s)a(s)f(u(s))ds + \frac{\lambda t}{1-\alpha\eta} \\ &> N_2 \left[\int_{1/4}^{3/4} g(s)a(s)u(s)ds + \frac{\alpha}{2(1-\alpha\eta)} \int_{1/4}^{3/4} G(\eta,s)a(s)u(s)ds \right] \\ &\geq N_2 \left[\frac{1}{4} \int_{1/4}^{3/4} g(s)a(s)ds + \frac{\alpha}{8(1-\alpha\eta)} \int_{1/4}^{3/4} G(\eta,s)a(s)ds \right] \|u\| \\ &= \|u\|. \end{aligned}$$

Thus we have

$$\|Tu\| > \|u\|, \quad u \in \partial K_{r_2}. \tag{3.7}$$

Now, since $f_\infty = 0$, there exists $\tilde{R} > 0$ such that

$$f(u) \leq \frac{N_1}{2} u, \quad u \in [\tilde{R}, \infty).$$

We consider two cases: f is bounded or f is unbounded.
Case 1 : Suppose that f is bounded, there exists $M > 0$ satisfy $f(u) \leq M$ for all $u \in [0, \infty)$. We choose $R_2 > \max\{2r_2, \frac{2M}{N_1}, \frac{2\lambda}{1-\alpha\eta}\}$ and $u \in \partial K_{R_2}$, then for $t \in [0,1]$, we have

$$\begin{aligned} Tu(t) &= \int_0^1 G(t,s)a(s)f(u(s))ds + \frac{\alpha t}{2(1-\alpha\eta)} \int_0^1 G(\eta,s)a(s)f(u(s))ds + \frac{\lambda t}{1-\alpha\eta} \\ &\leq M \int_0^1 g(s)a(s)ds + \frac{\alpha M}{2(1-\alpha\eta)} \int_0^1 g(s)a(s)ds + \frac{\lambda}{1-\alpha\eta} \end{aligned}$$

$$< \frac{M}{N_1} + \frac{R_2}{2} < \|u\|.$$

Case 2 : Suppose that f is unbounded. Let $R_2 > \max\{2r_2, \tilde{R}, \frac{2\lambda}{1-\alpha\eta}\}$ such that

$$f(u) \le f(R_2) \quad \text{for } 0 < u \le R_2.$$

(We are able to do this since f is unbounded.) Then for $u \in \partial K_{R_2}$, $t \in [0,1]$, we have

$$\begin{aligned} Tu(t) &= \int_0^1 G(t,s)a(s)f(u(s))ds + \frac{\alpha t}{2(1-\alpha\eta)}\int_0^1 G(\eta,s)a(s)f(u(s))ds + \frac{\lambda t}{1-\alpha\eta} \\ &\le \int_0^1 g(s)a(s)f(R_2)ds + \frac{\alpha}{2(1-\alpha\eta)}\int_0^1 g(s)a(s)f(R_2)ds + \frac{\lambda}{1-\alpha\eta} \\ &< \frac{N_1}{2}\left[\int_0^1 g(s)a(s)ds + \frac{\alpha}{2(1-\alpha\eta)}\int_0^1 g(s)a(s)ds\right]R_2 + \frac{R_2}{2} \\ &= \frac{R_2}{2} + \frac{R_2}{2} = \|u\|. \end{aligned}$$

Therefore, in either case we get

$$\|Tu\| < \|u\| \quad \text{for } u \in \partial K_{R_2}. \tag{3.8}$$

Therefore, by Lemma 2.5, (3.7), (3.8) and the property of fixed-point index, we obtain

$$i(T, K_{R_2}\backslash\overline{K_{r_2}}, K) = 1.$$

That is the operator T has at least one fixed point $u \in K_{R_2}\backslash\bar{K}_{r_2}$, which is a positive solution of BVP (1.1) and (1.2). □

Corollary 3.5. *Assume that $\lambda = 0$ hold, if the nonlinearity f is either superlinear or sublinear, then BVP* (1.1) *and* (1.2) *has at least one positive solution.*

Proof. The conclusion readily follows from Theorem 3.1 and Theorem 3.4. □

Example 3.6. Consider the following singular third-order three-point nonhomogeneous BVP

$$\begin{cases} u'''(t) + \frac{1}{t^q(1-t)}u^p(t) = 0, \quad 0 < t < 1, \\ u(0) = u''(0) = 0, \ u(1) - \frac{3}{2}u(\frac{1}{2}) = \lambda, \end{cases} \tag{3.9}$$

where $a(t) = \frac{1}{t^q(1-t)}$, $f(u) = u^p$, $\alpha = \frac{3}{2}$ and $\eta = \frac{1}{2}$, p and q are parameters which are positive and λ are nonnegative. If $p > 1$, $0 < q < 1$, then it is easy to verify that all conditions of Theorem 3.3 are satisfied, thus, there is $\Lambda > 0$ such that BVP (3.9) has at least one positive solution for $\lambda \in (0, \Lambda)$ and has no positive solution for $\lambda \in (\Lambda, \infty)$. If $0 < p, q < 1$, all conditions of Theorem 3.4 hold, then BVP (3.9) has at least one positive solution for $\lambda \in (0, \infty)$.

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