BOUNDARY VALUE PROBLEMS FOR SECOND ORDER IMPULSIVE FUNCTIONAL DIFFERENTIAL EQUATIONS

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ABSTRACT. This paper is concerned with the existence and approximation of solutions for second order impulsive functional differential equations with boundary value conditions. By establishing new comparison results and applying the monotone iterative technique, we obtain the sufficient conditions for the existence of extremal solutions.

AMS (MOS) Subject Classification. 34G25, 39J35, 45N05

1. INTRODUCTION

The theory of impulsive differential equations has been emerging as an important area of investigations in recent years since it is a basic tool to study some problems of biology, medicine, engineering, and physics(see [1], [2]). As an important branch, boundary value problems have drawn much attention. There are plenty of results on studying the boundary value problem of impulsive differential equations ([3]-[19]).

In this paper, we study the following boundary value problems

$$(1)\qquad \begin{cases} x''(t) = f(t, x(t), x(\theta(t))), \quad t \neq t_k,\ t \in J = [0, T], \\ \Delta x(t_k) = I_k(x(t_k)), \quad k = 1, 2, \ldots, m, \\ \Delta x'(t_k) = I_k^*(x'(t_k)), \\ x(0) = x(T) + \lambda_1 \int_0^T x(s)ds + a, \\ x'(0) = \lambda_2 x'(T) + \lambda_3 \int_0^T x(s)ds + b, \end{cases}$$

where $f \in C(J \times R \times R, R)$, $0 \leq \theta(t) \leq t$, $t \in J$. $0 = t_0 < t_1 < t_2 < \cdots < t_m < t_{m+1} = T$. $I_k, I_k^* \in C(R, R)$. $\Delta x(t_k) = x(t_k^+) - x(t_k^-), \Delta x'(t_k) = x'(t_k^+) - x'(t_k^-)$ for $k = 1, 2, \ldots, m$. $\lambda_1, \lambda_2, \lambda_3, a, b \in R$ and $\lambda_1, \lambda_3 \geq 0, 0 \leq \lambda_2 \leq 1$.

It is important to indicate that (1) includes a number of (impulsive) functional differential equations, (impulsive) functional differential equations with deviating arguments, (impulsive) functional differential equations with periodic boundary conditions studied by many authors as special cases, such as ([3]-[13]).

This paper is organized as follows: In section 2, we establish a new comparison principle. In section 3, after introducing the definitions of upper and lower solutions, we obtain existence of extremal solutions for (1) by using the method of upper and lower solutions and monotone iterative technique.

2. PRELIMINARIES AND COMPARISON PRINCIPLES

Let $PC(J) = \{x : J \to R; x(t)$ is continuous everywhere except for some t_k at which $x(t_k^+)$ and $x(t_k^-)$ exist, and $x(t_k^-) = x(t_k),\ k = 1, 2, \ldots, m\}$. $PC^1(J) = \{x \in PC(J) : x'(t)$ is continuous everywhere except for some t_k at which $x'(t_k^+)$ and $x'(t_k^-)$ exist, and $x'(t_k^-) = x'(t_k),\ k = 1, 2, \ldots, m\}$. $PC^2(J) = \{x \in PC^1(J) : x|_{(t_k,t_{k+1}]} \in C^2(t_k, t_{k+1}],\ k = 0, 1, \ldots m\}$. $PC(J)$ and $PC^1(J)$ are Banach spaces with the norms $\| x \|_{PC} = \sup\{|x(t)| :\ t \in J\}$ and $\| x \|_{PC^1} = \max\{\| x \|_{PC},\ \| x' \|_{PC}\}$.

A function $x \in PC^2(J)$ is called a solution of problem(1) if it satisfies (1).

The following comparison results and lemmas play an important role.

Lemma 2.1 ([1], P. 33–35) Suppose that the following conditions are satisfied

(A_0) the sequence $\{t_k\}$ satisfies $0 \le t_0 < t_1 < t_2 < \cdots$, with $\lim_{k\to\infty} t_k = \infty$.

(A_1) $m \in PC^1(R_+, R)$ and $m(t)$ is left continuous at t_k, $k = 1, 2 \ldots$.

(A_2) for $k = 1, 2 \ldots, t \ge t_0$

$$\begin{cases} m'(t) \le p(t)m(t) + q(t), & t \ne t_k, \\ m(t_k^+) \le d_k m(t_k) + b_k, & k = 1, 2 \ldots, m, \end{cases}$$

where $q, p \in C(R_+, R)$, $d_k \ge 0$ and b_k are real constants. Then

$$\begin{aligned} m(t) \le{}& m(t_0) \left(\prod_{t_0<t_k<t} d_k \right) \exp\left(\int_{t_0}^{t} p(s)ds \right) \\ &+ \sum_{t_0<t_k<t} \left(\prod_{t_k<t_j<t} d_j \right) \exp\left(\int_{t_k}^{t} p(s)ds \right) b_k \\ &+ \int_{t_0}^{t} \left(\prod_{s<t_k<t} d_k \right) \exp\left(\int_{s}^{t} p(\sigma)d\sigma \right) q(s)ds. \end{aligned}$$

Lemma 2.2 If $m \in PC^2(J)$ and

$$(2)\qquad \begin{cases} m''(t) \le -Mm(t) - Nm(\theta(t)), & t \ne t_k, it \in J = [0, T], \\ \Delta m(t_k) \le -L_k m(t_k), & k = 1, 2, \ldots, m, \\ \Delta m'(t_k) \le -L_k^* m'(t_k), & k = 1, 2, \ldots, m, \\ m'(0) \le pm'(T), \quad m(0) \le m(T), \end{cases}$$

where $M > 0$, $N \geq 0$, $0 \leq L_k < 1$, $0 < L_k^* < 1$ $(k = 1, 2, \dots m)$, $0 \leq p \leq 1$, such that

$$(3)\qquad (M+N)T\left\{p\left[1-p\prod_{k=1}^{m}(1-L_k^*)\right]^{-1}+\left(\prod_{k=1}^{m}(1-L_k^*)\right)^{-1}\right\}\int_0^T\prod_{s<t_k<T}(1-L_k)ds\leq\prod_{k=1}^{m}(1-L_k)^2.$$

Then $m(t) \leq 0$ for all $t \in J$.

Proof. The idea of the proof comes from [7], where similar comparison results are given if $p = 1$. Suppose, to the contrary,that there exists some $t \in [0, T]$ such that $m(t) > 0$. It is enough to consider the following two possible cases.

Case 1. There exists a $\bar{t} \in J$ such that $m(\bar{t}) > 0$ and $m(t) \geq 0$ for all $t \in [0,\ T]$. Then by (2) $m''(t) \leq 0$, for $t \neq t_k$ and $m'(t_k^+) \leq (1 - L_k^*)m'(t_k)$. Therefore, by lemma 2.1 we get $m'(t) \leq m'(0)\prod_{0<t_k<t}(1 - L_k^*)$. Hence by using (2) again, we have $m'(0) \leq pm'(T) \leq pm'(0)\prod_{k=1}^{m}(1 - L_k^*)$. Since $0 < L_k^* < 1$ $(k = 1, 2, \dots m)$, $0 \leq p \leq 1$, then $m'(0) \leq 0$. So $m'(t) \leq m'(0)\prod_{0<t_k<t}(1 - L_k^*) \leq 0$, which implies that $m(t)$ is non-increasing from $m(t_k^+) \leq (1 - L_k)m(t_k) \leq m(t_k)$. Therefore, we can obtain $m(T) \leq m(0)$. But by (2) we have $m(0) \leq m(T)$. So $m(t) = m(\bar{t}) = C > 0, \forall t \in J$. Hence $m''(t) \equiv 0$. However from (2) we have $m''(t) \leq -Mm(t) - Nm(\theta(t)) \leq -MC - NC < 0$, which is a contradiction.

Case 2. There exist $t_*,\ t^* \in J$ such that $m(t_*) < 0$ and $m(t^*) > 0$.Let $m(t_*) = \inf_{s\in J} m(s) = -\lambda,\ \lambda > 0,\ t_* \in (t_i, t_{i+1}]$. Set $t_* \neq t_i^+$ (if $t_* = t_i^+$, we may prove in the same way). In virtue of (2), we get the following inequalities:

$$\begin{cases} m''(t) \leq -Mm(t) - Nm(\theta(t)) \leq (M+N)\lambda,\ t \neq t_k,\ t \in J, \\ m'(t_k^+) \leq (1 - L_k^*)m'(t_k),\ k = 1, 2, \dots, m. \end{cases}$$

By lemma 2.1 we get

$$(4)\qquad m'(t) \leq m'(0)\prod_{0<t_k<t}(1 - L_k^*) + \int_0^t \prod_{s<t_k<t}(1 - L_k^*)(M+N)\lambda ds.$$

Let $t = T$, we obtain from (2)

$$\begin{aligned} m'(0) \leq pm'(T) &\leq pm'(0)\prod_{k=1}^{m}(1 - L_k^*) + p\lambda(M+N)\int_0^T \prod_{s<t_k<T}(1 - L_k^*)ds \\ &\leq pm'(0)\prod_{k=1}^{m}(1 - L_k^*) + p\lambda(M+N)T, \end{aligned}$$

which implies

$$(5)\qquad m'(0) \leq p\lambda(M+N)T[1 - p\prod_{k=1}^{m}(1 - L_k^*)]^{-1}.$$

By combining (4) and (5), we have

$$
\begin{aligned}
m'(t) &\le p\lambda(M+N)T[1-p\prod_{k=1}^{m}(1-L_k^*)]^{-1}\prod_{0<t_k<t}(1-L_k^*) \\
&\quad +\lambda(M+N)\int_0^t\prod_{s<t_k<t}(1-L_k^*)ds \\
&\le \lambda(M+N)\prod_{0<t_k<t}(1-L_k^*)\{pT[1-p\prod_{k=1}^{m}(1-L_k^*)]^{-1} \\
&\quad +\int_0^t\prod_{s<t_k<t}(1-L_k^*)ds/\prod_{0<t_k<t}(1-L_k^*)\} \\
&\le \lambda(M+N)\{pT[1-p\prod_{k=1}^{m}(1-L_k^*)]^{-1}+\int_0^T\prod_{s<t_k<T}(1-L_k^*)ds/\prod_{k=1}^{m}(1-L_k^*)\} \\
&\le \lambda(M+N)T\{p[1-p\prod_{k=1}^{m}(1-L_k^*)]^{-1}+(\prod_{k=1}^{m}(1-L_k^*))^{-1}\}.
\end{aligned}
$$

Using the inequalities $m(t_k^+)\le(1-L_k)m(t_k)$, we obtain for $t\in[t_*,T]$

$$
\begin{aligned}
m(t) &\le m(t_*)\textstyle\prod_{t_*<t_k<t}(1-L_k)+\lambda(M+N)T\{p[1-p\prod_{k=1}^{m}(1-L_k^*)]^{-1} \\
&\quad +(\textstyle\prod_{k=1}^{m}(1-L_k^*))^{-1}\}\int_{t_*}^{t}\prod_{s<t_k<t}(1-L_k)ds.
\end{aligned} \tag{6}
$$

If $t^*>t_*$, then we get from the inequality above

$$
\begin{aligned}
0<m(t^*) &\le -\lambda\prod_{t_*<t_k<t^*}(1-L_k)+\lambda(M+N)T\{p[1-p\prod_{k=1}^{m}(1-L_k^*)]^{-1} \\
&\quad +(\prod_{k=1}^{m}(1-L_k^*))^{-1}\}\int_{t_*}^{t^*}\prod_{s<t_k<t^*}(1-L_k)ds.
\end{aligned}
$$

So

$$
\begin{aligned}
\prod_{k=1}^{m}(1-L_k)^2/\int_0^T\prod_{s<t_k<T}(1-L_k)ds &\le \prod_{k=1}^{m}(1-L_k)/\int_0^T\prod_{s<t_k<T}(1-L_k)ds \\
&\le \prod_{t_*<t_k<t^*}(1-L_k)/\int_{t_*}^{t^*}\prod_{s<t_k<t^*}(1-L_k)ds \\
&< (M+N)T\{p[1-p\prod_{k=1}^{m}(1-L_k^*)]^{-1} \\
&\quad +(\prod_{k=1}^{m}(1-L_k^*))^{-1}\},
\end{aligned}
$$

which is a contradiction with (3).

If $t^*<t_*$, we may assume $t_*\in(t_i,t_{i+1}]$, $t^*\in(t_j,t_{j+1}]$, $0\le j\le i$. Like (6), we have

by lemma 2.1

$$\begin{aligned} m(t^*) &\leq m(0)\prod_{0<t_k<t^*}(1-L_k)+\lambda(M+N)T\{p[1-p\prod_{k=1}^{m}(1-L_k^*)]^{-1} \\ &\quad +(\prod_{k=1}^{m}(1-L_k^*))^{-1}\}\int_0^{t^*}\prod_{s<t_k<t^*}(1-L_k)ds \\ &= m(0)\prod_{k=1}^{j}(1-L_k)+\lambda(M+N)T\{p[1-p\prod_{k=1}^{m}(1-L_k^*)]^{-1} \\ &\quad +(\prod_{k=1}^{m}(1-L_k^*))^{-1}\}\int_0^{t^*}\prod_{s<t_k<t^*}(1-L_k)ds. \end{aligned}$$

On the other hand, from (2) and (6) with $t=T$, we obtain

$$\begin{aligned} m(0) &\leq m(T)\leq m(t_*)\prod_{t_*<t_k<T}(1-L_k)+\lambda(M+N)T\{p[1-p\prod_{k=1}^{m}(1-L_k^*)]^{-1} \\ &\quad +(\prod_{k=1}^{m}(1-L_k^*))^{-1}\}\int_{t_*}^{T}\prod_{s<t_k<T}(1-L_k)ds \\ &= -\lambda\prod_{k=i+1}^{m}(1-L_k)+\lambda(M+N)T\{p[1-p\prod_{k=1}^{m}(1-L_k^*)]^{-1} \\ &\quad +(\prod_{k=1}^{m}(1-L_k^*))^{-1}\}\int_{t_*}^{T}\prod_{s<t_k<T}(1-L_k)ds. \end{aligned}$$

From the two inequalities above, we obtain

$$\begin{aligned} 0<m(t^*) &\leq -\lambda\prod_{k=i+1}^{m}(1-L_k)\prod_{k=1}^{j}(1-L_k) \\ &\quad +\lambda(M+N)T\{p[1-p\prod_{k=1}^{m}(1-L_k^*)]^{-1}+(\prod_{k=1}^{m}(1-L_k^*))^{-1}\} \\ &\quad \times\prod_{k=1}^{j}(1-L_k)\int_{t_*}^{T}\prod_{s<t_k<T}(1-L_k)ds \\ &\quad +\lambda(M+N)T\{p[1-p\prod_{k=1}^{m}(1-L_k^*)]^{-1} \\ &\quad +(\prod_{k=1}^{m}(1-L_k^*))^{-1}\}\int_0^{t^*}\prod_{s<t_k<t^*}(1-L_k)ds. \end{aligned}$$

So

$$\prod_{k=i+1}^{m}(1-L_k)\prod_{k=1}^{j}(1-L_k)$$

$$< \ (M+N)T\{p[1-p\prod_{k=1}^{m}(1-L_k^*)]^{-1}+(\prod_{k=1}^{m}(1-L_k^*))^{-1}\}$$
$$\times[\int_0^{t^*}\prod_{s<t_k<t^*}(1-L_k)ds+\prod_{k=1}^{j}(1-L_k)\int_{t_*}^{T}\prod_{s<t_k<T}(1-L_k)ds]. \ (7)$$

Multiplying both sides of (7) by $\prod_{k=j+1}^{m}(1-L_k)$, we have

$$\begin{aligned}
&\prod_{k=i+1}^{m}(1-L_k)\prod_{k=1}^{m}(1-L_k)\\
< \ & (M+N)T\{p[1-p\prod_{k=1}^{m}(1-L_k^*)]^{-1}+(\prod_{k=1}^{m}(1-L_k^*))^{-1}\}\\
&\times[\prod_{k=j+1}^{m}(1-L_k)\int_0^{t^*}\prod_{s<t_k<t^*}(1-L_k)ds+\prod_{k=1}^{m}(1-L_k)\int_{t_*}^{T}\prod_{s<t_k<T}(1-L_k)ds]\\
\leq \ & (M+N)T\{p[1-p\prod_{k=1}^{m}(1-L_k^*)]^{-1}+(\prod_{k=1}^{m}(1-L_k^*))^{-1}\}\\
&\times[\int_0^{t^*}\prod_{s<t_k<T}(1-L_k)ds+\int_{t_*}^{T}\prod_{s<t_k<T}(1-L_k)ds]\\
= \ & (M+N)T\{p[1-p\prod_{k=1}^{m}(1-L_k^*)]^{-1}+(\prod_{k=1}^{m}(1-L_k^*))^{-1}\}\int_0^{T}\prod_{s<t_k<T}(1-L_k)ds.
\end{aligned}$$

Therefore,

$$\prod_{k=1}^{m}(1-L_k)^2<(M+N)T\{p[1-p\prod_{k=1}^{m}(1-L_k^*)]^{-1}+(\prod_{k=1}^{m}(1-L_k^*))^{-1}\}\int_0^{T}\prod_{s<t_k<T}(1-L_k)ds,$$

which is a contradiction with (3). Hence $m(t)\leq 0, \ t\in[0,T]$. The proof is complete.

Let us consider the following linear problem:

$$(8) \quad \begin{cases}
x''(t)+Mx(t)+Nx(\theta(t))=\sigma(t), \quad t\neq t_k, \ t\in J=[0,T],\\
\Delta x(t_k)=-L_kx(t_k)+a_k, \quad k=1,2,\ldots,m,\\
\Delta x'(t_k)=-L_k^*x'(t_k)+b_k, \quad k=1,2,\ldots,m,\\
x(0)=x(T)+c,\\
x'(0)=\lambda_2x'(T)+d,
\end{cases}$$

where $a_k, \ b_k, \ c, \ d, \ \lambda_2$ are constants.

Lemma 2.3 $x\in PC^2(J)$ is a solution of (8) if and only if $x\in PC^1(J)$ is a solution of the following impulsive integral equation. Furthermore, the linear problem (8) is uniquely solvable.

$$\begin{aligned}
(Ax)(t) \ = \ & q(t)+\int_0^{T}G_1(t,s)[Nx(\theta(s))-\sigma(s)]ds+\sum_{0<t_k<T}G_2(t,t_k)(-L_kx(t_k)+a_k)\\
&-\sum_{0<t_k<T}G_1(t,t_k)(-L_k^*x'(t_k)+b_k), \ (9)
\end{aligned}$$

where

$$q(t)=\frac{1}{mn}(mc\cos mt+d\sin mt-\lambda_2 mc\cos m(T-t)+d\sin m(T-t)),$$

$m=\sqrt{M},\ n=[1+\lambda_2-(1+\lambda_2)\cos mT]$,

$$G_1(t,s)=\frac{1}{mn}\begin{cases}-\sin m(t-s)-\frac{1}{2}(1+\lambda_2)\sin m(T-t+s)\\ -\frac{1}{2}(1-\lambda_2)\sin m(T-t-s),\ 0\le s<t\le T,\\ \lambda_2\sin m(t-s)-\frac{1}{2}(1+\lambda_2)\sin m(T+t-s)\\ -\frac{1}{2}(1-\lambda_2)\sin m(T-t-s),\ 0\le t\le s\le T,\end{cases}$$

$$G_2(t,s)=\frac{1}{n}\begin{cases}\cos m(t-s)-\frac{1}{2}(1+\lambda_2)\cos m(T-t+s)\\ -\frac{1}{2}(1-\lambda_2)\cos m(T-t-s),\ 0\le s<t\le T,\\ -\lambda_2\cos m(t-s)+\frac{1}{2}(1+\lambda_2)\cos m(T+t-s)\\ -\frac{1}{2}(1-\lambda_2)\cos m(T-t-s),\ 0\le t\le s\le T.\end{cases}$$

Proof. Suppose that $x(t)$ is a solution of (9), then

$$\begin{aligned}x'(t) &= q'(t)+\int_0^T G_{1t}(t,s)[Nx(\theta(s))-\sigma(s)]ds+\sum_{0<t_k<T}G_{2t}(t,t_k)(-L_kx(t_k)+a_k)\\ &\quad -\sum_{0<t_k<T}G_{1t}(t,t_k)(-L_k^*x'(t_k)+b_k)),\end{aligned}$$

where

$$\begin{aligned}G_{1t}(t,s) &= -\frac{1}{n}\begin{cases}\cos m(t-s)-\frac{1}{2}(1+\lambda_2)\cos m(T-t+s)\\ -\frac{1}{2}(1-\lambda_2)\cos m(T-t-s),\ 0\le s<t\le T,\\ -\lambda_2\cos m(t-s)+\frac{1}{2}(1+\lambda_2)\cos m(T+t-s)\\ -\frac{1}{2}(1-\lambda_2)\cos m(T-t-s),\ 0\le t\le s\le T\end{cases}\\ &= -G_2(t,s),\end{aligned}$$

$$\begin{aligned}G_{2t}(t,s) &= \frac{m}{n}\begin{cases}-\sin m(t-s)-\frac{1}{2}(1+\lambda_2)\sin m(T-t+s)\\ -\frac{1}{2}(1-\lambda_2)\sin m(T-t-s),\ 0\le s<t\le T,\\ \lambda_2\sin m(t-s)-\frac{1}{2}(1+\lambda_2)\sin m(T+t-s)\\ -\frac{1}{2}(1-\lambda_2)\sin m(T-t-s),\ 0\le t\le s\le T\end{cases}\\ &= MG_1(t,s).\end{aligned}$$

Therefore, we have

$$\begin{aligned}x''(t) &= q''(t)+\int_0^T G_{1tt}(t,s)[Nx(\theta(s))-\sigma(s)]ds\\ &\quad +\sum_{0<t_k<T}G_{2tt}(t,t_k)(-L_kx(t_k)+a_k)\\ &\quad -\sum_{0<t_k<T}G_{1tt}(t,t_k)(-L_k^*x'(t_k)+b_k))-Nx(\theta(t))+\sigma(t),\end{aligned}$$

where

$$G_{1tt}(t,s) = -G_{2t}(t,s) = -MG_1(t,s),$$

$$G_{2tt}(t,s) = MG_{1t}(t,s) = -MG_2(t,s),$$

$$q''(t) = -\frac{m^2c}{n}\cos mt - \frac{dm}{n}\sin mt + \frac{\lambda_2 m^2 c}{n}\sin m(T-t) - \frac{dm}{n}\cos m(T-t) = -Mq(t).$$

Hence

$$x''(t) + Mx(t) + Nx(\theta(t)) = \sigma(t).$$

By directly computing, we get

$$\begin{aligned}\Delta x(t_k) &= \Delta \sum_{0<t_k<T} [G_2(t,t_k)(-L_k x(t_k) + a_k) - G_1(t,t_k)(-L_k^* x'(t_k) + b_k)]|_{t=t_k} \\ &= -L_k x(t_k) + a_k,\end{aligned}$$

$$\begin{aligned}\Delta x'(t_k) &= \Delta \sum_{0<t_k<T} [G_{2t}(t,t_k)(-L_k x(t_k) + a_k) - G_{1t}(t,t_k)(-L_k^* x'(t_k) + b_k)]|_{t=t_k} \\ &= -L_k^* x'(t_k) + b_k.\end{aligned}$$

It also easy to see $G_1(0,s) = G_1(T,s),\ G_2(0,s) = G_2(T,s),\ q(0) = q(T) + c,\ q'(0) = \lambda_2 q'(T) + d$. So $x(0) = x(T) + c,\ x'(0) = \lambda_2 x'(T) + d$.

Now, we prove existence of solution for problem (9), Consider the operators $A : PC^1(J) \to PC^1(J)$

$$\begin{aligned}x(t) &= q(t) + \int_0^T G_1(t,s)[Nx(\theta(s)) - \sigma(s)]ds + \sum_{0<t_k<T} G_2(t,t_k)(-L_k x(t_k) + a_k) \\ &\quad - \sum_{0<t_k<T} G_1(t,t_k)(-L_k^* x'(t_k) + b_k).\end{aligned}$$

Similar to the proof of Lemma 3.2 in [8], we can prove that operator A is continuous and compact. Applying the Schauder fixed-point theorem, we obtain existence of a fixed point $x \in PC^1(J)$ for A, so $x \in PC^1(J)$ is a solution of (8).

Next, we show that the solution of (8) is unique. Suppose that $x_1(t),\ x_2(t) \in PC^2(J)$ are two solutions of (8). Let $m(t) = x_1(t) - x_2(t)$, then

$$\begin{aligned}m''(t) &= [\sigma(t) - Mx_1(t) - Nx_1(\theta(t))] - [\sigma(t) - Mx_2(t) - Nx_2(\theta(t))] \\ &= -M[x_1(t) - x_2(t)] - N[x_1(\theta(t)) - x_2(\theta(t))] \\ &= -Mm(t) - Nm(\theta(t)),\end{aligned}$$

$$\begin{aligned}\Delta m(t_k) &= [-L_k x_1(t_k) + a_k] - [-L_k x_2(t_k) + a_k] \\ &= -L_k[x_1(t_k) - x_2(t_k)] = -L_k m(t_k),\end{aligned}$$

$$\begin{aligned}\Delta m'(t_k) &= [-L_k^* x_1'(t_k) + b_k] - [-L_k^* x_2'(t_k) + b_k] \\ &= -L_k^*[x_1'(t_k) - x_2'(t_k)] = -L_k^* m'(t_k),\end{aligned}$$

$$m(0) = x_1(0) - x_2(0) = x_1(T) + c - (x_2(T) + c) = x_1(T) - x_2(T) = m(T),$$

$$m'(0) = x_1'(0) - x_2'(0) = \lambda_2 x_1'(T) + d - (\lambda_2 x_2'(T) + d) = \lambda_2(x_1'(T) - x_2'(T)) = \lambda_2 m'(T).$$

By lemma 2.2, we have $m(t) \leq 0$, that is $x_1(t) \leq x_2(t)$ on J. Analogously, we can show that $x_2(t) \leq x_1(t)$ on J. Therefore $x_1(t) = x_2(t)$ on J. It means that (8) has a unique solution. The proof is complete.

3. MAIN RESULTS

In this section, we shall use monotone iterative technique to obtain the existence of extremal solutions of (1).

Definition 3.1 A function $\alpha \in PC^2(J)$ is called a lower solution of (1) if

$$\begin{cases} \alpha''(t) \leq f(t, \alpha(t), \alpha(\theta(t))), \ t \neq t_k, \ t \in J = [0, T], \\ \Delta\alpha(t_k) \leq I_k(\alpha(t_k)), \ k = 1, 2, \ldots, m, \\ \Delta\alpha'(t_k) \leq I_k^*(\alpha'(t_k)), \ k = 1, 2, \ldots, m, \\ \alpha(0) \leq \alpha(T) + \lambda_1 \int_0^T \alpha(s)ds + a, \\ \alpha'(0) \leq \lambda_2\alpha'(T) + \lambda_3 \int_0^T \alpha(s)ds + b. \end{cases}$$

Analogously, $\beta \in PC^2(J)$ is called an upper solution of (1) if

$$\begin{cases} \beta''(t) \geq f(t, \beta(t), \beta(\theta(t))), \ t \neq t_k, \ t \in J = [0, T], \\ \Delta\beta(t_k) \geq I_k(\beta(t_k)), \ k = 1, 2, \ldots, m, \\ \Delta\beta'(t_k) \geq I_k^*(\beta'(t_k)), \ k = 1, 2, \ldots, m, \\ \beta(0) \geq \beta(T) + \lambda_1 \int_0^T \beta(s)ds + a, \\ \beta'(0) \geq \lambda_2\beta'(T) + \lambda_3 \int_0^T \beta(s)ds + b. \end{cases}$$

In the sequel, we need the following assumptions.

(H1) α, β are lower and upper solutions of (1) such that $\alpha \leq \beta$.

(H2) There exist constants $M > 0$ and $N \geq 0$, such that

$$f(t, x, y) - f(t, \bar{x}, \bar{y}) \geq -M(x - \bar{x}) - N(y - \bar{y}),$$

wherever $\alpha(t) \leq \bar{x}(t) \leq x(t) \leq \beta(t)$ $\alpha(\theta(t)) \leq \bar{y}(t) \leq y(t) \leq \beta(\theta(t))$.

(H3) There exist constants $0 \leq L_k < 1$, $0 < L_k^* < 1$ for $k = 1, 2, \ldots m$, such that

$$I_k(x(t_k)) - I_k(y(t_k)) \geq -L_k(x(t_k) - y(t_k)),$$

$$I_k^*(x'(t_k)) - I_k^*(y'(t_k)) = -L_k^*(x'(t_k) - y'(t_k)),$$

wherever $\alpha(t_k) \leq y(t_k) \leq x(t_k) \leq \beta(t_k)$, $k = 1, 2, \ldots m$.

Let $[\alpha(t), \beta(t)] = \{\ x \in PC^1(J) :\ \alpha(t) \leq x(t) \leq \beta(t)\}$ for all $t \in J$.

Now we are in the position to establish the main results of this paper.

Lemma 3.1 Let (H1), (H2), (H3) and inequality (3) hold. Then there exist $y(t), z(t)$

satisfying the following two linear impulsive functional differential equations,respectively,

$$\begin{cases} y''(t)+My(t)+Ny(\theta(t))=f(t,\alpha(t),\alpha(\theta(t)))+M\alpha(t)+N\alpha(\theta(t)),\\ \Delta y(t_k)=-L_k y(t_k)+I_k(\alpha(t_k))+L_k\alpha(t_k),\ k=1,2,\dots,m,\\ \Delta y'(t_k)=-L_k^* y'(t_k)+I_k^*(\alpha'(t_k))+L_k^*\alpha'(t_k),\ k=1,2,\dots,m,\\ y(0)=y(T)+\lambda_1\int_0^T\alpha(s)ds+a,\\ y'(0)=\lambda_2 y'(T)+\lambda_3\int_0^T\alpha(s)ds+b,\end{cases}$$

and

$$\begin{cases} z''(t)+Mz(t)+Nz(\theta(t))=f(t,\beta(t),\beta(\theta(t)))+M\beta(t)+N\beta(\theta(t)),\\ \Delta z(t_k)=-L_k z(t_k)+I_k(\beta(t_k))+L_k\beta(t_k),\ k=1,2,\dots,m,\\ \Delta z'(t_k)=-L_k^* z'(t_k)+I_k^*(\beta'(t_k))+L_k^*\beta'(t_k),\ k=1,2,\dots,m,\\ z(0)=z(T)+\lambda_1\int_0^T\beta(s)ds+a,\\ z'(0)=\lambda_2 z'(T)+\lambda_3\int_0^T\beta(s)ds+b,\end{cases}$$

such that $\alpha(t)\le y(t)\le z(t)\le\beta(t),\ t\in J$, and $y(t),z(t)$ are also lower and upper solutions of (1), respectively.

Proof. Let $m(t)=\alpha(t)-y(t)$, then

$$\begin{aligned} m''(t) &\le f(t,\alpha(t),\alpha(\theta(t)))-[f(t,\alpha(t),\alpha(\theta(t)))\\ &\quad +M\alpha(t)+N\alpha(\theta(t))-My(t)-Ny(\theta(t))]\\ &= -M(\alpha(t)-y(t))-N(\alpha(\theta(t))-y(\theta(t)))=-Mm(t)-Nm(\theta(t)),\end{aligned}$$

$$\begin{aligned}\Delta m(t_k) &\le I_k(\alpha(t_k))-[-L_k y(t_k)+I_k(\alpha(t_k))+L_k\alpha(t_k)]\\ &= -L_k[\alpha(t_k)-y(t_k)]=-L_k m(t_k),\end{aligned}$$

$$\begin{aligned}\Delta m'(t_k) &\le I_k^*(\alpha'(t_k))-[-L_k^* y'(t_k)+I_k^*(\alpha'(t_k))+L_k^*\alpha'(t_k)]\\ &= -L_k^*[\alpha'(t_k)-y'(t_k)]=-L_k^* m'(t_k),\end{aligned}$$

$$\begin{aligned} m(0) &\le \alpha(T)+\lambda_1\int_0^T\alpha(s)ds+a-(y(T)+\lambda_1\int_0^T\alpha(s)ds+a)\\ &= \alpha(T)-y(T)=m(T),\end{aligned}$$

$$\begin{aligned} m'(0) &\le \lambda_2\alpha'(T)+\lambda_3\int_0^T\alpha(s)ds+b-(\lambda_2 y'(T)+\lambda_3\int_0^T\alpha(s)ds+b)\\ &= \lambda_2\alpha'(T)-\lambda_2 y'(T)=\lambda_2 m'(T).\end{aligned}$$

By lemma 2.2, we get $m(t)\le 0$ for $t\in J$, that is, $\alpha(t)\le y(t)$.
Similarly, we can prove that $z(t)\le\beta(t)$.
Now we suppose that $m(t)=y(t)-z(t)$, then we have

$$\begin{aligned} m''(t) &= [f(t,\alpha(t),\alpha(\theta(t)))+M\alpha(t)+N\alpha(\theta(t))-My(t)-Ny(\theta(t))]\\ &\quad -[f(t,\beta(t),\beta(\theta(t)))+M\beta(t)+N\beta(\theta(t))-Mz(t)-Nz(\theta(t))]\\ &\le -M[y(t)-z(t)]-N[y(\theta(t))-z(\theta(t))]=-Mm(t)-Nm(\theta(t)),\end{aligned}$$

$$\begin{aligned}\Delta m(t_k) &= -L_k(y(t_k)-z(t_k))+I_k(\alpha(t_k))+L_k\alpha(t_k)-I_k(\beta(t_k))-L_k\beta(t_k)]\\ &\le -L_k(y(t_k)-z(t_k))=-L_km(t_k),\end{aligned}$$

$$\begin{aligned}\Delta m'(t_k) &= -L_k^*(y'(t_k)-z'(t_k))+I_k^*(\alpha'(t_k))+L_k^*\alpha'(t_k)-I_k^*(\beta'(t_k))-L_k^*\beta'(t_k)\\ &\le -L_k^*(y'(t_k)-z'(t_k))=-L_k^*m'(t_k),\end{aligned}$$

$$\begin{aligned}m(0) &= (y(T)+\lambda_1\int_0^T\alpha(s)ds+a)-(z(T)+\lambda_1\int_0^T\beta(s)ds+a)\\ &= y(T)-z(T)+\lambda_1\int_0^T(\alpha(s)ds-\beta(s))ds\le m(T),\end{aligned}$$

$$\begin{aligned}m'(0) &= \lambda_2y'(T)+\lambda_3\int_0^T\alpha(s)ds+b-(\lambda_2z'(T)+\lambda_3\int_0^T\beta(s)ds+b)\\ &= \lambda_2y'(T)-\lambda_2z'(T)+\lambda_3\int_0^T(\alpha(s)ds-\beta(s))ds\le\lambda_2m'(T).\end{aligned}$$

By lemma 2.2, we get $m(t)\le 0$ for $t\in J$, that is $y(t)\le z(t)$. So $\alpha(t)\le y(t)\le z(t)\le\beta(t),\ t\in J$. Now, from $(H2)$ and $(H3)$, we obtain

$$\begin{aligned}y''(t) &= [f(t,\alpha(t),\alpha(\theta(t)))+M\alpha(t)+N\alpha(\theta(t))-My(t)-Ny(\theta(t))]\\ &\le [f(t,y(t),y(\theta(t)))+My(t)+Ny(\theta(t))-My(t)-Ny(\theta(t))]\\ &= f(t,y(t),y(\theta(t))),\end{aligned}$$

$$\begin{aligned}\Delta y(t_k) &= -L_ky(t_k)+I_k(\alpha(t_k))+L_k\alpha(t_k)\\ &\le -L_ky(t_k)+I_k(y(t_k))+L_ky(t_k)=I_k(y(t_k)),\end{aligned}$$

$$\begin{aligned}y'(t_k) &= -L_k^*y'(t_k)+I_k^*(\alpha'(t_k))+L_k^*\alpha'(t_k)\\ &= -L_k^*y'(t_k)+I_k^*(y'(t_k))+L_k^*y'(t_k)=I_k^*(y'(t_k)),\end{aligned}$$

$$y(0)=y(T)+\lambda_1\int_0^T\alpha(s)ds+a\le y(T)+\lambda_1\int_0^Ty(s)ds+a,$$

$$y'(0)=\lambda_2y'(T)+\lambda_3\int_0^T\alpha(s)ds+b\le\lambda_2y'(T)+\lambda_3\int_0^Ty(s)ds+b.$$

So $y(t)$ is a lower solution of (1). Similarly, we can prove that $z(t)$ is an upper solution of (1). The proof is complete.

Theorem 3.2 Let (H1), (H2), (H3) and inequality (3) hold. Then there exist monotone sequences $\{\alpha_n\},\{\beta_n\}\subset[\alpha,\beta]$ with $\alpha=\alpha_0\le\alpha_1\le\cdots\alpha_n\le\cdots\le\beta_n\le\cdots\le\beta_1\le\beta_0=\beta$ such that $\lim_{n\to\infty}\alpha_n=x_*,\ \lim_{n\to\infty}\beta_n=x^*$ uniformly on J. Moreover,

x_*, x^* are minimal and maximal solution of (1) in $[\alpha, \beta]$, respectively.

Proof. For $n = 1, 2, \ldots$ we suppose that

$$\begin{cases} \alpha_n''(t) + M\alpha_n(t) + N\alpha_n(\theta(t)) = f(t, \alpha_{n-1}(t), \alpha_{n-1}(\theta(t))) + M\alpha_{n-1}(t) + N\alpha_{n-1}(\theta(t)), \\ \Delta\alpha_n(t_k) = -L_k\alpha_n(t_k) + I_k(\alpha_{n-1}(t_k)) + L_k\alpha_{n-1}(t_k),\ k = 1, 2, \ldots, m, \\ \Delta\alpha_n'(t_k) = -L_k^*\alpha_n'(t_k) + I_k^*(\alpha_{n-1}'(t_k)) + L_k^*\alpha_{n-1}'(t_k),\ k = 1, 2, \ldots, m, \\ \alpha_n(0) = \alpha_n(T) + \lambda_1 \int_0^T \alpha_{n-1}(s)ds + a, \\ \alpha_n'(0) = \lambda_2\alpha_n'(T) + \lambda_3 \int_0^T \alpha_{n-1}(s)ds + b, \end{cases}$$

and

$$\begin{cases} \beta_n''(t) + M\beta_n(t) + N\beta_n(\theta(t)) = f(t, \beta_{n-1}(t), \beta_{n-1}(\theta(t))) + M\beta_{n-1}(t) + N\beta_{n-1}(\theta(t)), \\ \Delta\beta_n(t_k) = -L_k\beta_n(t_k) + I_k(\beta_{n-1}(t_k)) + L_k\beta_{n-1}(t_k),\ k = 1, 2, \ldots, m, \\ \Delta\beta_n'(t_k) = -L_k^*\beta_n'(t_k) + I_k^*(\beta_{n-1}'(t_k)) + L_k^*\beta_{n-1}'(t_k),\ k = 1, 2, \ldots, m, \\ \beta_n(0) = \beta_n(T) + \lambda_1 \int_0^T \beta_{n-1}(s)ds + a, \\ \beta_n'(0) = \lambda_2\beta_n'(T) + \lambda_3 \int_0^T \beta_{n-1}(s)ds + b. \end{cases}$$

In view of Lemma 3.1, we easily get the two monotone sequences $\{\alpha_n\}, \{\beta_n\} \subset [\alpha, \beta]$ such that $\alpha = \alpha_0 \le \alpha_1 \le \cdots \alpha_n \le \cdots \le \beta_n \le \cdots \le \beta_1 \le \beta_0 = \beta$. And $\lim_{n\to\infty} \alpha_n = x_*$, $\lim_{n\to\infty} \beta_n = x^*$ uniformly on J. Moreover, $x_*(t)$, $x^*(t)$ are solutions of (1) in $[\alpha(t), \beta(t)]$.

To prove that $x_*(t)$, $x^*(t)$ are extremal solutions of (1), let $x \in [\alpha, \beta]$ be any solution of (1),that is,

$$\begin{cases} x''(t) = f(t, x(t), x(\theta(t))),\ t \ne t_k,\ t \in J = [0, T], \\ \Delta x(t_k) = I_k(x(t_k)),\ k = 1, 2, \ldots, m, \\ \Delta x'(t_k) = I_k^*(x'(t_k)),\ k = 1, 2, \ldots, m \\ x(0) = x(T) + \lambda_1 \int_0^T x(s)ds + a, \\ x'(0) = \lambda_2 x'(T) + \lambda_3 \int_0^T x(s)ds + b. \end{cases}$$

Suppose that there exists a positive integer n such that $\alpha_n(t) \le x \le \beta_n(t)$ on J. Then, let $m(t) = \alpha_{n+1}(t) - x(t)$, we have,

$$\begin{aligned} m''(t) &= [f(t, \alpha_n(t), \alpha_n(\theta(t))) + M\alpha_n(t) + N\alpha_n(\theta(t)) \\ &\quad -M\alpha_{n+1}(t) - N\alpha_{n+1}(\theta(t))] - f(t, x(t), x(\theta(t))) \\ &\le -M(\alpha_{n+1}(t) - x(t)) - N(\alpha_{n+1}(\theta(t)) - x(\theta(t))) = -Mm(t) - Nm(\theta(t)), \end{aligned}$$

$$\begin{aligned} \Delta m(t_k) &= [-L_k\alpha_{n+1}(t_k) + I_k(\alpha_n(t_k)) + L_k\alpha_n(t_k)] - I_k(x(t_k)) \\ &\le -L_k[\alpha_{n+1}(t_k) - x(t_k)] = -L_k m(t_k), \end{aligned}$$

$$\begin{aligned} \Delta m'(t_k) &= [-L_k^*\alpha_{n+1}'(t_k) + I_k^*(\alpha_n'(t_k)) + L_k^*\alpha_n'(t_k)] - I_k^*(x'(t_k)) \\ &\le -L_k^*[\alpha_{n+1}'(t_k) - x'(t_k)] = -L_k^* m'(t_k), \end{aligned}$$

$$\begin{aligned} m(0) &= \alpha_{n+1}(T)+\lambda_1\int_0^T \alpha_n(s)ds + a - (x(T)+\lambda_1\int_0^T x(s)ds + a) \\ &= \alpha_{n+1}(T)-x(T)+\lambda_1(\int_0^T(\alpha_n(s)-x(s))ds) \le \alpha_{n+1}(T)-x(T)=m(T), \end{aligned}$$

$$\begin{aligned} m'(0) = &= \lambda_2\alpha'_{n+1}(T)+\lambda_3\int_0^T \alpha_n(s)ds + b - (\lambda_2 x'(T)+\lambda_3\int_0^T x(s)ds + b) \\ &= \lambda_2\alpha'_{n+1}(T)-\lambda_2 x'(T)+\lambda_3(\int_0^T(\alpha_n(s)-x(s))ds) \le \lambda_2 m'(T). \end{aligned}$$

By lemma 2.2, $m(t) \le 0$ on J, i.e, $\alpha_{n+1}(t) \le x$ on J. Similarly we obtain $x \le \beta_{n+1}(t)$ on J. Since $\alpha_0 \le x(t) \le \beta_0$ on J, by induction we get $\alpha_n(t) \le x \le \beta_n(t)$ on J for every n. Therefore, $x_*(t) \le x(t) \le x^*(t)$ on J by taking $n \to \infty$. The proof is complete.

We conclude with a simple example which can be treated by the methods developed above.

Example. Consider the following boundary value problem:

$$(10) \qquad \begin{cases} x''(t) = -\frac{1}{100}x(t) - \frac{1}{50}e^{-t}x(\frac{1}{2}t),\ t\in[0,1], t\ne\frac{1}{2}, \\ \Delta x(\frac{1}{2}) = -\frac{5}{6}x(\frac{1}{2}), \\ \Delta x'(\frac{1}{2}) = -\frac{3}{4}x'(\frac{1}{2}), \\ x(0) = x(1)+\frac{1}{100}\int_0^1 x(s)ds+\frac{1}{10}, \\ x'(0) = \frac{1}{2}x'(1)+\frac{1}{10}\int_0^1 x(s)ds+\frac{1}{5}, \end{cases}$$

Put

$$\alpha(t)=0, t\in[0,1], \beta(t) = \begin{cases} t+1, t\in[0,\frac{1}{2}], \\ \frac{1}{2}t, t\in(\frac{1}{2},1]. \end{cases}$$

Obviously, $\alpha(t)$ is a lower solution and $\beta(t)$ is an upper solutions of (10). Let

$$f(t,x,y) = -\frac{1}{100}x - \frac{1}{50}e^{-t}y,\ M=\frac{1}{100}, N=\frac{1}{50}, L_1=\frac{5}{6}, L_1^*=\frac{3}{4}, \lambda_2=\frac{1}{2}.$$

We have

$$f(t,x,y)-f(t,\bar{x},\bar{y}) = -\frac{1}{100}(x-\bar{x}) - \frac{1}{50}e^{-t}(y-\bar{y}),$$

$$I_1(x(\frac{1}{2})) - I_1(y(\frac{1}{2})) = -\frac{5}{6}(x(\frac{1}{2})-y(\frac{1}{2})),$$

$$I_1^*(x'(\frac{1}{2})) - I_1^*(y'(\frac{1}{2})) = -\frac{3}{4}(x'(\frac{1}{2})-y'(\frac{1}{2})),$$

$$\begin{aligned} &(M+N)T\{\lambda_2[1-\lambda_2\prod_{k=1}^m(1-L_k^*)]^{-1} + (\prod_{k=1}^m(1-L_k^*))^{-1}\}\int_0^T \prod_{s<t_k<T}(1-L_k)ds \\ &= (\frac{4}{7}+4)\frac{1}{200} = \frac{4}{175} \le \prod_{k=1}^m(1-L_k)^2 = \frac{1}{36}. \end{aligned}$$

Hence, (10) satisfies all conditions of Theorem 3.1. It follows that there exist monotone sequences $\{\alpha_n\}, \{\beta_n\}$ uniformly converging to the extremal solutions of (10) in $[\alpha, \beta]$.

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