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A STABILITY RESULT IN A MEMORY-TYPE TIMOSHENKO SYSTEM

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ABSTRACT. In this paper we consider the following Timoshenko system

$$\varphi_{tt} - (\varphi_x + \psi)_x = 0, \quad (0,1) \times \mathbb{R}_+$$

$$\psi_{tt} - \psi_{xx} + \varphi_x + \psi + \int_0^t g(t-\tau)\psi_{xx}(\tau)d\tau = 0, \quad (0,1) \times \mathbb{R}_+$$

with Dirichlet boundary conditions where g is a positive nonincreasing function. We establish a generalized stability result for this system.

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1. INTRODUCTION

Timoshenko [1] gave the following system of coupled hyperbolic equations

$$\rho u_{tt} = (K(u_x - \varphi))_x, \quad \text{in } (0,L) \times (0,+\infty)$$

$$I_\rho \varphi_{tt} = (EI\varphi_x)_x + K(u_x - \varphi), \text{ in } (0,L) \times (0,+\infty), \tag{1.1}$$

as a simple model describing the transverse vibration of a beam, where t denotes the time variable and x is the space variable along the beam of length L, in its equilibrium configuration, u is the transverse displacement of the beam and φ is the rotation angle of the filament of the beam. The coefficients ρ, I_ρ, E, I and K are respectively the density (the mass per unit length), the polar moment of inertia of a cross section, Young's modulus of elasticity, the moment of inertia of a cross section, and the shear modulus.

System (1.1) has been studied by many mathematicians and results concerning existence and asymptotic behavior have been established. Kim and Renardy [2] considered (1.1) together with two linear boundary conditions of the form

$$\begin{aligned} K\varphi(L,t) - K\frac{\partial u}{\partial x}(L,t) &= \alpha\frac{\partial u}{\partial t}(L,t), \quad \forall t \geq 0 \\ EI\frac{\partial \varphi}{\partial x}(L,t) &= -\beta\frac{\partial \varphi}{\partial t}(L,t), \quad \forall t \geq 0 \end{aligned}$$

and used the multiplier techniques to establish an exponential decay result for the energy of (1.1). They also provided numerical estimates to the eigenvalues of the operator associated with system (1.1). An analogous result was also established by Feng *et al.* [3], where the stabilization of vibrations in a Timoshenko system was studied. Raposo *et al.* [4] studied (1.1) with homogeneous Dirichlet boundary conditions and two linear frictional dampings. Precisely, they looked into the following system

$$\begin{aligned} &\rho_1 u_{tt} - K(u_x - \varphi)_x + u_t = 0, \text{ in } (0,L)\times(0,+\infty) \\ &\rho_2\varphi_{tt} - b\varphi_{xx} + K(u_x-\varphi) + \varphi_t = 0, \text{ in } (0,L)\times(0,+\infty) \\ &u(0,t) = u(L,t) = \varphi(0,t) = \varphi(L,t) = 0, \quad \forall t > 0 \end{aligned} \tag{1.2}$$

and proved that the energy associated with (1.2) decays exponentially. To obtain their result, they used a method developed by Liu and Zheng [5]. This method is different from the usual ones such as the classical energy method. It mainly uses the semigroup theory. Soufyane and Wehbe [6] showed that it is possible to stabilize uniformly (1.1) by using a unique locally distributed feedback. So, they considered

$$\begin{aligned} &\rho u_{tt} = (K(u_x - \varphi))_x, \text{ in } (0,L)\times(0,+\infty) \\ &I_\rho\varphi_{tt} = (EI\varphi_x)_x + K(u_x-\varphi) - b\varphi_t, \text{ in } (0,L)\times(0,+\infty) \\ &u(0,t) = u(L,t) = \varphi(0,t) = \varphi(L,t) = 0, \quad \forall t > 0, \end{aligned} \tag{1.3}$$

where b is a positive and continuous function, which satisfies

$$b(x) \geq b_0 > 0, \quad \forall\, x \in [a_0, a_1] \subset [0, L].$$

In fact, they proved that the uniform stability of (1.3) holds if and only if the wave speeds are equal $\left(\frac{K}{\rho} = \frac{EI}{I_\rho}\right)$; otherwise only the asymptotic stability has been proved. This result improves earlier ones by Soufyane [7] and Shi and Feng [8], where an exponential decay of the solution energy of (1.1) together, with two locally distributed feedbacks, had been proved. Xu and Yung [9] studied a system of Timoshenko beams with pointwise feedback controls, sought information about the eigenvalues and eigenfunctions of the system, and used this information to examine the stability of the system. Ammar-Khodja *et al.* [10] considered a linear Timoshenko-type system with memory of the form

$$\begin{aligned} &\rho_1\varphi_{tt} - K(\varphi_x + \psi)_x = 0 \\ &\rho_2\psi_{tt} - b\psi_{xx} + \int_0^t g(t-s)\psi_{xx}(s)ds + K(\varphi_x + \psi) = 0 \end{aligned} \tag{1.4}$$

in $(0,L)\times(0,+\infty)$, together with homogeneous boundary conditions. They used the multiplier techniques and proved that the system is uniformly stable if and only if the wave speeds are equal $\left(\frac{K}{\rho_1} = \frac{b}{\rho_2}\right)$ and g decays uniformly. Precisely, they proved an exponential decay if g decays in an exponential rate and polynomially if g decays in a polynomial rate. They also required some extra technical conditions on both g' and g'' to obtain their result. The feedback of memory type has also been used by Santos [11].

He considered a Timoshenko system and showed that the presence of two feedbacks of memory type at a portion of the boundary stabilizes the system uniformly. He also obtained the rate of decay of the energy, which is exactly the rate of decay of the relaxation functions. Shi and Feng [12] investigated a nonuniform Timoshenko beam and showed that, under some locally distributed controls, the vibration of the beam decays exponentially. We refer the reader to [13], [14], and [15] for more results.

In the present work we are concerned with

$$(1.5)\qquad \begin{cases} \varphi_{tt}-(\varphi_x+\psi)_x=0, \quad (0,1)\times \mathbb{R}_+ \\ \psi_{tt}-\psi_{xx}+\varphi_x+\psi+\int_0^t g(t-\tau)\psi_{xx}(\tau)d\tau=0, \quad (0,1)\times \mathbb{R}_+ \\ \varphi(0,t)=\varphi(1,t)=\psi(0,t)=\psi(1,t)=0,\ t\geq 0 \\ \varphi(x,0)=\varphi_0(x),\ \varphi_t(x,0)=\varphi_1(x),\ x\in(0,1) \\ \psi(x,0)=\psi_0(x),\ \psi_t(x,0)=\psi_1(x),\ x\in(0,1). \end{cases}$$

Our aim in this work is to establish a generalized stability result for system (1.5). We should note here that we do not loose the generality by taking ρ_1, ρ_2, K, b, appeared in (1.4), to be equal to one and our argument also works for $\rho_1/\rho_2 = K/b$. The paper is organized as follows. In section 2, we present some notations and material needed for our work. In section 3, we prove several technical lemmas. The statement and the proof of our main result will be given in section 4. Finally, we give some comments and remarks.

2. **PRELIMINARIES**

In this section we present some material needed in the proof of our main result. We use the standard Lebesgue space $L^2(0,1)$ and the Sobolev space $H_0^1(0,1)$ with their usual scalar products and norms. We will use c, throughout this paper, to denote a generic positive constant.

For the relaxation function g we assume

(H1) $g:\mathbb{R}_+\to\mathbb{R}_+$ is a differentiable function such that

$$g(0)>0, \qquad 1-\int_0^\infty g(s)ds=l>0.$$

(H2) There exists a nonincreasing differentiable function $\xi:\mathbb{R}_+\to\mathbb{R}_+$ satisfying

$$g'(t)\leq -\xi(t)g(t),\ t\geq 0$$

For completeness we state, without proof, an existence and regularity result.

Proposition 2.1. *Let* (φ_0,φ_1), $(\psi_0,\psi_1)\in H_0^1(0,1)\times L^2(0,1)$ *be given. Assume that (H1) and (H2) are satisfied, then problem* (1.5) *has a unique global (weak) solution*

$$\varphi,\ \psi\in C(\mathbb{R}_+;H_0^1(0,1))\cap C^1(\mathbb{R}_+;L^2(0,1)).$$

Moreover, if

$$(\varphi_0, \varphi_1),\, (\psi_0, \psi_1) \in (H^2(0,1) \cap H_0^1(0,1)) \times H_0^1(0,1)$$

then the solution satisfies

$$\varphi,\, \psi \in C(\mathbb{R}_+; H^2(0,1) \cap H_0^1(0,1)) \cap C^1(\mathbb{R}_+; H_0^1(0,1)) \cap C^2(\mathbb{R}_+; L^2(0,1)).$$

Remark 2.1. This result can be proved using standard arguments such as the semigroup method or the Galerkin method.

Now, we introduce the energy functional

$$E(t) := \frac{1}{2}\int_0^1 \left(\varphi_t^2 + \psi_t^2 + (1 - \int_0^t g(s)ds)\psi_x^2 + (\varphi_x + \psi)^2\right) dx + \frac{1}{2}(g \circ \psi_x), \tag{2.1}$$

where, for all $v \in L^2(0,1)$,

$$(g \circ v)(t) = \int_0^1 \int_0^t g(t-s)(v(t) - v(s))^2 ds\, dx.$$

3. TECHNICAL LEMMAS

In this section we establish several lemmas needed to prove our main result.

Lemma 3.1. *Let (φ, ψ) be the solution of (1.5). Then the energy functional satisfies*

$$E'(t) = -\frac{1}{2}g(t)\int_0^1 \psi_x^2 dx + \frac{1}{2}(g' \circ \psi_x) \le 0. \tag{3.1}$$

Proof. By multiplying equations in (1.5) by φ_t and ψ_t respectively and integrating over $(0,1)$, using integration by parts, hypotheses (H1)-(H2) and some manipulations, we obtain (3.1) for any regular solution. This equality remains valid for weak solutions by a simple density argument. □

Using Cauchy-Schwartz and Poincaré's inequalities, the proof of the following lemma is immediate.

Lemma 3.2. *There exists $c > 0$ such that, for all $u \in H_0^1(0,1)$,*

$$\int_0^1 \left(\int_0^t g(t-s)(u(t) - u(s))ds\right)^2 dx \le c(g \circ u_x)(t).$$

Now we are going to construct a Lyapunov functional $\mathcal{L}$ equivalent to E. For this, we define several functionals which allow us to obtain the needed estimates.

Lemma 3.3. *Under the assumptions (H1) and (H2), the functional I defined by*

$$I(t) := -\int_0^1 \psi_t \int_0^t g(t-s)(\psi(t) - \psi(s))ds\, dx$$

satisfies, along the solution, the estimate

$$I'(t) \le -(\int_0^t g(s)ds - \delta)\int_0^1 \psi_t^2 dx + \delta \int_0^1 (\varphi_x + \psi)^2 dx \tag{3.2}$$

$$+c\delta\int_0^1 \psi_x^2 dx + c(\delta+\frac{1}{\delta})(g\circ\psi_x)(t) - \frac{c}{\delta}g'\circ\psi_x$$

for all $\delta > 0$.

Proof. Direct computations, using (1.5), yield

$$\begin{aligned} I'(t) &= -\int_0^1 \psi_t \int_0^t g'(t-s)(\psi(t)-\psi(s))ds\,dx - (\int_0^t g(s)ds)\int_0^1 \psi_t^2 dx \\ &\quad - \int_0^1 [\psi_{xx} - \int_0^t g(t-s)\psi_{xx}(s)ds - \varphi_x - \psi]\int_0^t g(t-s)(\psi(t)-\psi(s))ds\,dx \\ &= -\int_0^1 \psi_t \int_0^t g'(t-s)(\psi(t)-\psi(s))ds\,dx - (\int_0^t g(s)ds)\int_0^1 \psi_t^2 dx \\ &\quad + \int_0^1 \psi_x \int_0^t g(t-s)(\psi_x(t)-\psi_x(s))ds\,dx \\ &\quad + \int_0^1 (\varphi_x+\psi)\int_0^t g(t-s)(\psi(t)-\psi(s))ds\,dx \\ &\quad - \int_0^1 (\int_0^t g(t-s)\psi_x(s)ds)(\int_0^t g(t-s)(\psi_x(t)-\psi_x(s))ds)dx \end{aligned}$$

We now estimate the terms in the right side of the above equality as follows.

By using Young's inequality and Lemma 3.2 [for $(-g')$] we obtain, for all $\delta > 0$,

$$-\int_0^1 \psi_t \int_0^t g'(t-s)(\psi(t)-\psi(s))ds\,dx \le \delta\int_0^1 \psi_t^2 dx - \frac{c}{\delta}(g'\circ\psi_x).$$

Similarly, we have

$$\int_0^1 \psi_x \int_0^t g(t-s)(\psi_x(t)-\psi_x(s))ds\,dx \le \delta\int_0^1 \psi_x^2 dx + \frac{c}{\delta}g\circ\psi_x,$$

$$\int_0^1 (\varphi_x+\psi)\int_0^t g(t-s)(\psi(t)-\psi(s))ds\,dx \le \delta\int_0^1 (\varphi_x+\psi)^2 dx + \frac{c}{\delta}g\circ\psi_x,$$

and

$$\begin{aligned} &-\int_0^1 (\int_0^t g(t-s)\psi_x(s)ds)(\int_0^t g(t-s)(\psi_x(t)-\psi_x(s))ds)dx \\ &\le \delta\int_0^1 \left(\int_0^t g(t-s)(\psi_x(s)-\psi_x(t)+\psi_x(t))ds\right)^2 dx \\ &\quad + \frac{c}{\delta}\int_0^1 \left(\int_0^t g(t-s)(\psi_x(t)-\psi_x(s))ds\right)^2 dx \\ &\le 2\delta\int_0^1 \psi_x^2 \left(\int_0^t g(s)ds\right)^2 dx + (2\delta+\frac{c}{\delta})\int_0^1 \left(\int_0^t g(t-s)(\psi_x(t)-\psi_x(s))ds\right)^2 dx \\ &\le c\delta\int_0^1 \psi_x^2 dx + c(\delta+\frac{1}{\delta})g\circ\psi_x \end{aligned}$$

By combining all the above estimates, the assertion of Lemma 3.3 is proved. □

Lemma 3.4. *Under the assumptions (H1) and (H2), the functional K_1 defined by*

$$K_1(t) := -\int_0^1 (\psi\psi_t + \varphi\varphi_t)dx$$

satisfies, along the solution, the estimate

$$K_1'(t) \le -\int_0^1 (\psi_t^2 + \varphi_t^2)dx + \int_0^1 (\psi + \varphi_x)^2 dx + c\int_0^1 \psi_x^2 dx + cg \circ \psi_x \tag{3.3}$$

Proof. By exploiting equations (1.5) and repeating the same procedure as in above, we have

$$\begin{aligned} K_1'(t) &= -\int_0^1 (\psi_t^2 + \varphi_t^2)dx - \int_0^1 \varphi(\psi_x + \varphi_{xx})dx \\ &\quad - \int_0^1 \psi[\psi_{xx} - \int_0^t g(t-s)(\psi_x(s))_x ds - \varphi_x - \psi]dx \\ &= -\int_0^1 (\psi_t^2 + \varphi_t^2)dx + \int_0^1 \psi_x^2 dx - \int_0^1 \psi_x \left[\int_0^t g(t-s)\psi_x(s)ds\right] dx \\ &\quad + \int_0^1 (\psi + \varphi_x)^2 dx \\ &\le -\int_0^1 (\psi_t^2 + \varphi_t^2)dx + \int_0^1 (\psi + \varphi_x)^2 dx + c\int_0^1 \psi_x^2 dx + cg \circ \psi_x \end{aligned}$$

This completes the proof. □

Lemma 3.5. *Assume that (H1) and (H2) hold. Then, for any $0 < \varepsilon < 1$, the functional K_2 defined by*

$$K_2(t) := \int_0^1 \psi_t(\psi + \varphi_x)dx + \int_0^1 \psi_x\varphi_t dx - \int_0^1 \varphi_t \int_0^t g(t-s)\psi_x(s)ds\, dx$$

satisfies, along the solution, the estimate

$$\begin{aligned} K_2'(t) \le &\left[\left(\psi_x - \int_0^t g(t-s)\psi_x(s)ds\right)\varphi_x\right]_{x=0}^{x=1} + \varepsilon c\int_0^1 \varphi_t^2 dx \\ &- \int_0^1 (\psi + \varphi_x)^2 dx + \frac{c}{\varepsilon}\int_0^1 \psi_x^2 dx + \int_0^1 \psi_t^2 dx - \frac{c}{\varepsilon} g' \circ \psi_x. \end{aligned} \tag{3.4}$$

Proof. Using equations (1.5) and integrating by parts yield

$$\begin{aligned} K_2'(t) = &\int_0^1 (\varphi_x + \psi)[\psi_{xx} - \int_0^t g(t-s)\psi_{xx}(s)ds - \varphi_x - \psi]dx \\ &+ \int_0^1 (\varphi_{xt} + \psi_t)\psi_t dx + \int_0^1 \psi_{xt}\varphi_t dx + \int_0^1 \psi_x(\varphi_x + \psi)_x dx \\ &- \int_0^1 (\varphi_x + \psi)_x \int_0^t g(t-s)\psi_x(s)ds\, dx \\ &- \int_0^1 \varphi_t \left(g(0)\psi_x + \int_0^t g'(t-s)\psi_x(s)ds\right) dx \end{aligned}$$

$$= \left[\left(\psi_x - \int_0^t g(t-s)\psi_x(s)ds \right) \varphi_x \right]_{x=0}^{x=1} - \int_0^1 (\psi + \varphi_x)^2 dx + \int_0^1 \psi_t^2 dx$$
$$- g(t) \int_0^1 \psi_x \varphi_t dx - \int_0^1 \varphi_t \int_0^t g'(t-s)(\psi_x(s) - \psi_x(t)) ds\, dx$$

By using Young's inequality, (3.4) is established. □

Lemma 3.6. *Assume that (H1) and (H2) hold. Let $m \in C^1([0,1])$ be a function satisfying $m(0) = -m(1) = 2$. Then there exists $c > 0$ such that, for any $0 < \varepsilon < 1$, the functional K_3 defined by*

$$K_3(t) := \frac{1}{4\varepsilon} \int_0^1 m(x)\psi_t \left(\psi_x - \int_0^t g(t-s)\psi_x(s)ds \right) dx + \varepsilon \int_0^1 m(x)\varphi_t \varphi_x dx$$

satisfies, along the solution, the estimate

$$\begin{aligned} K_3'(t) \le & -\frac{1}{4\varepsilon} \left[\left(\psi_x(1,t) - \int_0^t g(t-s)\psi_x(1,s)ds \right)^2 \right. \\ & \left. + \left(\psi_x(0,t) - \int_0^t g(t-s)\psi_x(0,s)ds \right)^2 \right] \\ & - \varepsilon \left(\varphi_x^2(1,t) + \varphi_x^2(0,t) \right) + (\frac{1}{4} + \varepsilon c) \int_0^1 (\psi + \varphi_x)^2 dx + \varepsilon c \int_0^1 \varphi_t^2 dx \\ & + \frac{c}{\varepsilon^2} \left(\int_0^1 \psi_x^2 dx + g \circ \psi_x \right) + \frac{c}{\varepsilon} \int_0^1 \psi_t^2 dx - \frac{c}{\varepsilon} (g' \circ \psi_x) \end{aligned} \tag{3.5}$$

Proof. Using equations (1.5) and integrating by parts yield

$$\begin{aligned} K_3'(t) = & \frac{1}{4\varepsilon} \int_0^1 m(x) \left(\psi_x - \int_0^t g(t-s)\psi_x(s)ds \right)_x \left(\psi_x - \int_0^t g(t-s)\psi_x(s)ds \right) dx \\ & - \frac{1}{4\varepsilon} \int_0^1 m(x) \left(\psi_x - \int_0^t g(t-s)\psi_x(s)ds \right) (\varphi_x + \psi) dx \\ & + \frac{1}{4\varepsilon} \int_0^1 m(x)\psi_t \left(\psi_{xt} - g(0)\psi_x - \int_0^t g'(t-s)\psi_x(s)ds \right) dx \\ & + \varepsilon \int_0^1 m(x)\psi_x \varphi_x dx + \varepsilon \int_0^1 m(x)\varphi_{xx}\varphi_x dx + \varepsilon \int_0^1 m(x)\varphi_t \varphi_{xt} dx \\ = & \frac{1}{4\varepsilon} \left[- \left([\psi_x(1,t) - \int_0^t g(t-s)\psi_x(1,s)ds]^2 + [\psi_x(0,t) - \int_0^t g(t-s)\psi_x(0,s)ds]^2 \right) \right. \\ & - \frac{1}{2} \int_0^1 m'(x) \left(\psi_x - \int_0^t g(t-s)\psi_x(s)ds \right)^2 dx \\ & - \int_0^1 m(x) \left(\psi_x - \int_0^t g(t-s)\psi_x(s)ds \right) (\varphi_x + \psi) dx - \frac{1}{2} \int_0^1 m'(x)\psi_t^2 dx \\ & \left. + \int_0^1 m(x)\psi_t \left(\int_0^t g'(t-s)(\psi_x(t) - \psi_x(s))ds \right) dx - g(t) \int_0^1 m(x)\psi_x \psi_t dx \right] \end{aligned}$$

$$+\varepsilon\left[\int_0^1 m(x)\psi_x\varphi_x dx-\left(\varphi_x^2(1,t)+\varphi_x^2(0,t)\right)\right.$$
$$\left.-\frac{1}{2}\int_0^1 m'(x)\varphi_x^2dx-\frac{1}{2}\int_0^1 m'(x)\varphi_t^2dx\right]$$

By using Young's and Poincaré's inequalities, Lemma 3.2,and the fact that

$$\varphi_x^2\leq 2(\psi+\varphi_x)^2+2\psi^2$$

we obtain (3.5). □

Lemma 3.7. *Assume that (H1) and (H2) hold. Then, after fixing ε small enough, the functional K defined by*

$$K(t):=3c\varepsilon K_1(t)+K_2(t)+K_3(t)$$

satisfies, along the solution, the estimate

(3.6)
$$K'(t)\leq-\frac{1}{2}\int_0^1(\psi+\varphi_x)^2dx-\tau\int_0^1\varphi_t^2dx+c\int_0^1\psi_t^2dx+c\int_0^1\psi_x^2dx+cg\circ\psi_x-cg'\circ\psi_x$$

where $\tau=c\varepsilon$.

Proof. By using Lemma 3.4, Lemma 3.5, Lemma 3.6, and the fact that

$$\left(\psi_x-\int_0^t g(t-s)\psi_x(s)ds\right)\varphi_x\leq\varepsilon\varphi_x^2+\frac{1}{4\varepsilon}\left(\psi_x-\int_0^t g(t-s)\psi_x(s)ds\right)^2,$$

we obtain (3.6). □

As in [10], we use the multiplier w which solves

$$-w_{xx}=\psi_x,\quad w(0)=w(1)=0. \tag{3.7}$$

Lemma 3.8. *The solution of* (3.7) *satisfies*

$$\int_0^1 w_x^2dx\leq\int_0^1\psi^2dx$$

and

$$\int_0^1 w_t^2dx\leq\int_0^1\psi_t^2dx.$$

Proof. We multiply equation (3.7) by w, integrate by parts, and use the Cauchy-Schwarz inequality, to get

$$\int_0^1 w_x^2dx\leq\int_0^1\psi^2dx.$$

Next, we differentiate (3.7) with respect to t to obtain, by similar calculations,

$$\int_0^1 w_{xt}^2dx\leq\int_0^1\psi_t^2dx.$$

Poincaré's inequality, then yields

$$\int_0^1 w_t^2 dx \le \int_0^1 \psi_t^2 dx.$$

This completes the proof of Lemma 3.8. □

Lemma 3.9. *Assume that (H1) and (H2) hold. Then, the functional J defined by*

$$J(t) := \int_0^1 (\psi\psi_t + w\varphi_t) dx$$

satisfies, along the solution, the estimate

$$J'(t) \le -\frac{l}{2}\int_0^1 \psi_x^2 dx + \frac{c}{\varepsilon_0}\int_0^1 \psi_t^2 dx + \varepsilon_0 \int_0^1 \varphi_t^2 dx + cg \circ \psi_x \tag{3.8}$$

for any $0 < \varepsilon_0 < l$ (l is defined in (H1)).

Proof. Using equations (1.5), integrating by parts, and Young's inequality, we get

$$\begin{aligned} J'(t) &= \int_0^1 (\psi_t^2 - \psi_x^2) dx + \int_0^1 \psi_x \int_0^t g(t-s)\psi_x(s) ds\, dx \\ &\quad - \int_0^1 \psi(\psi + \varphi_x) dx + \int_0^1 w(\psi_x + \varphi_{xx}) dx + \int_0^1 w_t \varphi_t dx \\ &\le \int_0^1 \psi_t^2 dx - \frac{l}{2}\int_0^1 \psi_x^2 dx + cg \circ \psi_x \\ &\quad + \int_0^1 (w_x^2 - \psi^2) dx + \varepsilon_0 \int_0^1 \varphi_t^2 dx + \frac{1}{4\varepsilon_0}\int_0^1 w_t^2 dx \end{aligned}$$

Then Poincaré's inequality and Lemma 3.8 give the desired result. □

4. GENERALIZED STABILITY

We are now ready to state and prove our main result.

Theorem 4.1. *Let $(\varphi_0, \varphi_1), (\psi_0, \psi_1) \in H_0^1(0,1) \times L^2(0,1)$ be given. Assume that (H1) and (H2) are satisfied, then there exist two positive constants c and ω, for which the solution of problem (1.5) satisfies, for t large,*

$$E(t) \le c e^{-\omega \int_0^t \xi(s) ds} \tag{4.1}$$

Proof. For $N_1, N_2, N_3 > 1$, let

$$\mathcal{L}(t) := N_1 E(t) + N_2 I(t) + N_3 J + K(t)$$

and let $g_0 = \int_0^{t_0} g(s) ds > 0$ for some fixed $t_0 > 0$. By combining (3.1), (3.2), (3.6), (3.8), and taking $\delta = 1/(4N_2)$ (in (3.2)), we arrive at

$$\begin{aligned} \mathcal{L}'(t) &\le -(\frac{lN_3}{2} - \frac{5}{4}c)\int_0^1 \psi_x^2 dx - (\tau - \varepsilon_0 N_3)\int_0^1 \varphi_t^2 dx \\ &\quad - (N_2 g_0 - \frac{1}{4} - c\frac{N_3}{\varepsilon_0} - c)\int_0^1 \psi_t^2 dx - \frac{1}{4}\int_0^1 (\psi + \varphi_x)^2 dx \end{aligned}$$

$$+ (4cN_2^2 + \frac{1}{4}c + cN_3 + c)(g \circ \psi_x)(t) + (\frac{N_1}{2} - 4cN_2^2 - c)(g' \circ \psi_x)(t) \tag{4.2}$$

for all $t \geq t_0$ and $0 < \varepsilon_0 < l$.

Now, we choose N_3 large enough so that

$$c_1 := (\frac{lN_3}{2} - \frac{5}{4}c) > 0,$$

then ε_0 small enough so that

$$c_2 := (\tau - \varepsilon_0 N_3) > 0$$

Next, we choose N_2 large enough so that

$$c_3 := (N_2 g_0 - \frac{1}{4} - c\frac{N_3}{\varepsilon_0} - c) > 0$$

Finally, we choose N_1 large enough so that

$$(\frac{N_1}{2} - 4cN_2^2 - c) > 0$$

Thus, (4.2) becomes

$$\mathcal{L}'(t) \leq -c_1 \int_0^1 \psi_x^2 dx - c_2 \int_0^1 \varphi_t^2 dx - c_3 \int_0^1 \psi_t^2 dx$$

$$-\frac{1}{4} \int_0^1 (\psi + \varphi_x)^2 dx + c(g \circ \psi_x)(t) \leq -kE(t) + c(g \circ \psi_x)(t), \tag{4.3}$$

for all $t \geq t_0$.

On the other hand, we can choose N_1 even larger (if needed) so that

$$\mathcal{L}(t) \sim E(t). \tag{4.4}$$

Therefore, by using (H2), (3.1), and (4.3), we obtain

$$\begin{aligned}
\xi(t)\mathcal{L}'(t) &\leq -k\xi(t)E(t) + c\xi(t)(g \circ \psi_x)(t) \\
&= -k\xi(t)E(t) + c\xi(t) \int_0^1 \int_0^t g(t-s)(\psi_x(t) - \psi_x(s))^2 ds\, dx \\
&\leq -k\xi(t)E(t) + c \int_0^1 \int_0^t \xi(t-s)g(t-s)(\psi_x(t) - \psi_x(s))^2 ds\, dx \\
&\leq -k\xi(t)E(t) - c \int_0^1 \int_0^t g'(t-s)(\psi_x(t) - \psi_x(s))^2 ds\, dx \\
&\leq -k\xi(t)E(t) - cE'(t), \qquad \forall t \geq t_0,
\end{aligned}$$

which gives

$$(\xi\mathcal{L} + cE)'(t) \leq -k\xi(t)E(t), \qquad \forall t \geq t_0.$$

Hence, using the fact that

$$F = \xi\mathcal{L} + cE \sim E, \tag{4.5}$$

we obtain, for some positive constant ω,

$$F'(t) \le -\omega\xi(t)F(t), \quad \forall t \ge t_0.$$

A simple integration over (t_0, t), leads to

$$F(t) \le F(t_0)e^{-\omega\int_{t_0}^{t}\xi(s)ds}, \quad \forall t \ge t_0.$$

Consequently, (4.1) is established by virtue of (4.5) and boundedness of E and ξ. $\square$

Examples. We give some examples to illustrate the energy decay rates obtained by Theorem 4.1.

(1) If $g(t) = ae^{-b(1+t)^p}$, $0 < p \le 1$, then $g'(t) = -\xi(t)g(t)$, where $\xi(t) = bp(1+t)^{p-1}$. For suitably chosen positive constants a and b, g satisfies (H1) and (H2), and (4.1) gives

$$E(t) \le ce^{-\omega b(1+t)^p}.$$

(2) If $g(t) = \frac{a}{(1+t)^q}$, $q > 1$, then

$$E(t) \le \frac{c}{(1+t)^{q\omega}}.$$

(3) If $g(t) = \frac{a}{(e+t)[\ln(e+t)]^s}$, $s > 1$, then

$$E(t) \le \frac{c}{[(e+t)[\ln(e+t)]^s]^{\omega}}.$$

The above three examples are included in the following more general one.

(4) For any nonincreasing function $g(t)$ which satisfies (H1) and $\frac{-g'}{g}$ is also nonincreasing, (4.1) gives

$$E(t) \le c[g(t)]^{\omega}.$$

5. FINAL COMMENTS

1) It has to be noted that our result allows large class of relaxation functions, and, in case $\int_0^\infty \xi(t)dt = +\infty$, (4.1) gives more general decay rate results for which the usual exponential and polynomial decay estimates are only special cases.

2) Our result is established under weaker conditions on g than those in [10]. Precisely, we do not require anything on g'' as in (1.6) and (1.7) of [10]. Also, to obtain exponential and polynomial decay results, the authors in [10] assumed that

$$-a_1 g^p(t) \le g'(t) \le -a_2 g^p(t) \tag{5.1}$$

for some $1 \le p < \frac{3}{2}$,while our result allows $1 \le p < 2$. In fact, (5.1) is a special case of (H2) with $\xi(t) = a_2 g^{p-1}(t)$, and since (5.1) yields

$$\frac{c_1}{1+t} \le g^{p-1}(t) \le \frac{c_2}{1+t}, \qquad 1 < p < 2,$$

then (4.1) gives an exponential decay for $p = 1$ and a polynomial decay for $1 < p < 2$.

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