GENERALIZED MONOTONE METHOD FOR ORDINARY AND CAPUTO FRACTIONAL DIFFERENTIAL EQUATIONS

AGHALAYA S VATSALA, M. SOWMYA, AND DONNA S. STUTSON

Department of Mathematics, University of Louisiana at Lafayette

Lafayette, Louisiana 70504 USA

R. V. College of Engineering, Bengaluru, Karnataka 560059, India

Department of Mathematics, Xavier University of Louisiana

New Orleans, Louisiana 70125 USA

ABSTRACT. The method of upper and lower solutions combined with monotone method is an efficient tool to compute the minimal and maximal solutions of a variety of nonlinear problems. In addition, the method of coupled lower and solutions combined with monotone iterative technique known as generalized monotone method is a very useful tool to compute the coupled minimal and maximal solutions of nonlinear dynamic systems. The advantage of generalized monotone method for fractional differential equations is that, it avoids the Mittag-Leffler function altogether. However, computing coupled lower and upper solution is nontrivial. This work is a survey on the known results in this direction.

AMS (MOS) Subject Classification. 34A08, 34A12.

1. INTRODUCTION

Mathematical models in various branches of science, engineering, finance, economics, etc will lead to the study of nonlinear dynamic systems namely nonlinear differential equations. From the modeling perspective dynamic systems with fractional derivatives are more applicable and useful models. See [4, 12, 10] for details. Study of fractional differential equations has gained importance in the past four decades due to its applications. Qualitative properties like existence and uniqueness for fractional differential equation uses some kind of fixed point theorem methods. However, these methods in general are not usually computational methods and in addition, they do not determine or guarantee the interval of existence. Monotone method combined with upper and lower solutions is an effective tool to show existence which is both theoretical and computational. The method provides natural sequences if the nonlinear function is increasing and alternate sequences if the nonlinear function is decreasing. Generalized monotone method using coupled lower and upper solutions provide natural sequences which converge uniformly and monotonically to coupled minimal and maximal solutions. The method is applicable when the nonlinear function is a sum of

increasing and decreasing terms. Further, if uniqueness conditions are satisfied, the coupled minimal and maximal solutions converges uniformly to the unique solution of the nonlinear problem. The advantage of the generalized monotone method for nonlinear fractional differential equations is that the iterates or the elements of the sequences constructed does not require the Mittag-Leffler function, which removes the computational complexity. However, the disadvantage of the generalized monotone method is finding nonconstant coupled lower and upper solutions on the interval of existence of the solution. The interval of existence can be determined by natural lower and upper solutions. Natural lower and upper solutions are relatively easy to compute since equilibrium solutions are in general natural lower and upper solutions. In this paper we survey the work done recently on the methodology of computing coupled lower and upper solutions to the desired interval of existence using natural lower and upper solutions. We have achieved this for scalar and system of fractional differential equations of order q, for $0 < q < 1$. The rate of convergence of these iterates are linear since the iterative scheme is just a modification of the generalized monotone method scheme. If we seek faster convergence method, then it leads to interesting open problems relative to the product rule of Mittag-Leffler functions.

So far, in literature, most models are differential equations with integer derivatives. A vast literature for the qualitative study of dynamic systems with integer order is available(see [5, 8]). However, the qualitative and quantitative study of fractional differential and integral equations has gained importance recently due to its applications. See [1, 3, 4, 6, 10, 12] for details of the study of fractional integral and differential equations of both Riemann Liouville and Caputo type. Many practical applications of fractional differential and integral equations have also been provided in the references of the monographs cited above. The qualitative study of fractional differential and integral equations of various types has been established in [2, 3, 4, 6, 11, 12, 15, 17]. Among the type of fractional dynamic systems, the study of Riemann Liouville and Caputo type of fractional dynamic systems has gained more importance.

2. PRELIMINARY AND AUXILLARY RESULTS

In this section, we recall known results, some definitions which are needed for our main results. Here, and throughout this work, we will consider Caputo fractional differential equations of order q, where $0 < q < 1$. All our results are true for $q = 1$, the integer derivative of order 1.

Definition 2.1. The Caputo fractional derivative of order q is given by:

$$^{c}D^{q}u(t) = \frac{1}{\Gamma(1-q)} \int_{0}^{t} (t-s)^{-q} u'(s) ds,$$

where $0 < q < 1$ and $\Gamma(q)$ is the Gamma function. Also, $u'(t)$ above is the first derivative of u with respect to t.

Although, in this work, we study Caputo fractional differential equations, our comparison result follows from the comparison result relative to the Riemann-Liouville derivative and the relation between Riemann-Liouville derivative and Caputo fractional derivative. Hence the next definition is for the Riemann-Liouville derivative.

Definition 2.2. Riemann-Liouville fractional derivative of order q with respect to t is defined by:

$$D^q u(t) = \frac{1}{\Gamma(1-q)} \frac{d}{dt} \int_0^t (t-s)^{-q} f(s) ds,$$

where $0 < q < 1$.

Here, and throughout this work, we will consider fractional differential equations of order q, where, $0 < q < 1$.

Consider the nonlinear Caputo fractional differential equation with initial condition of the form:

$$(2.1) \qquad\qquad {}^c D^q u(t) = f(t, u(t)), \quad u(0) = u_0,$$

where $f \in C[J \times \mathbb{R}, \mathbb{R}]$ and $J = [0, T]$. The integral representation of (2.1) is given by:

$$(2.2) \qquad\qquad u(t) = u_0 + \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} f(s, u(s)) ds.$$

The sequences we develop are always solutions of linear Caputo fractional differential equation. In order to compute the solution of the linear fractional differential equation with constant coefficients we need Mittag-Leffler function.

Definition 2.3. Mittag-Leffler function of two parameters q, r is given by

$$E_{q,r}(\lambda(t-t_0)^q) = \sum_{k=0}^{\infty} \frac{(\lambda(t-t_0)^q)^k}{\Gamma(qk+r)},$$

where $q, r > 0$. Also, for $t_0 = 0$ and $r = 1$, we get

$$E_{q,1}(\lambda t^q) = \sum_{k=0}^{\infty} \frac{(\lambda t^q)^k}{\Gamma(qk+1)},$$

where $q > 0$.

Also, consider the linear Caputo fractional differential equation,

$$(2.3) \qquad\qquad {}^c D^q u(t) = \lambda u(t) + f(t), \quad u(0) = u_0, \quad \text{on } J,$$

where $J = [0, T]$, λ is a constant and $f(t) \in C[J, \mathbb{R}]$. The solution of (2.3) exists and is unique. The explicit solution of (2.3) is given by:

$$(2.4) \qquad u(t) = u_0 E_{q,1}(\lambda t^q) + \int_0^t (t - s)^{q-1} E_{q,q}(\lambda(t - s)^q) f(s) ds.$$

See [4] for details. In particular, if $\lambda = 0$, the solution $u(t)$ is given by:

$$(2.5) \qquad u(t) = u_0 + \frac{1}{\Gamma(q)} \int_0^t (t - s)^{q-1} f(s) ds,$$

which is relatively easy to compute since we can avoid the computation of the Mittag-Leffler function.

Also we recall known results related to scalar Caputo nonlinear fractional differential equations of the following form.

$$(2.6) \qquad {}^c D^q u(t) = f(t, u) + g(t, u), \quad u(0) = u_0 \quad \text{on } J = [0, T],$$

where $0 < q < 1$. Results when $q = 1$ is proved in [14] and the references therein. Here $f, g \in C(J \times \mathbb{R}, \mathbb{R})$, $f(t, u)$ is non-decreasing in u on J and $g(t, u)$ is non-increasing in u on J. Note that the population model namely the logistic equation will be of the form (2.6).

In order to prove the comparison result relative to coupled lower and upper solutions of (2.6) we recall a basic lemma relative to the Riemann-Liouville fractional derivative. For that purpose, we need the following definition.

Definition 2.4. Let $p = 1 - q$, a function $m(t) \in C((0, T], R)$ is a C_p continuous function if $t^p m(t) \in C[J, R]$. The set of C_p continuous functions is denoted by $C_p[J, R]$. Further, given a function $m(t) \in C_p[J, R]$ we call the function $t^p m(t)$ the continuous extension of $m(t)$, on J.

Lemma 2.5. *Let* $m(t) \in C_p[J, \mathbb{R}]$ *(where* $J = [0, T]$*) be such that for some* $t_1 \in (0, T]$, $m(t_1) = 0$ *and* $m(t) \leq 0$, *on* $(0, T]$. *Then* $D^q m(t_1) \geq 0$.

Proof. See [2, 6] for details. Note that the above result has been proved in [2] without using the Hölder continuity assumption of $m(t)$. This improvement plays an important role in all iterative methods, like monotone method, generalized monotone methods etc.   □

The above lemma is true for Caputo derivative also, using the relation ${}^c D^q m(t) = D^q(m(t) - m(0))$ between the Caputo derivative and the Riemann-Liouville derivative. This is the version we will be using to prove our comparison results.

We recall the following known definitions which are needed for our main results.

Definition 2.6. The functions $v, w \in C^1(J, \mathbb{R})$ are called natural lower and upper solutions of (2.6) if :

$$\begin{cases} {}^cD^q v(t) \leq f(t, v) + g(t, v), & v(0) \leq u_0, \\ {}^cD^q w(t) \geq f(t, w) + g(t, w), & w(0) \geq u_0. \end{cases}$$

Definition 2.7. The functions $v, w \in C^1(J, \mathbb{R})$ are called coupled lower and upper solutions of (2.6) of type I if:

$$\begin{cases} {}^cD^q v(t) \leq f(t, v) + g(t, w), & v(0) \leq u_0, \\ {}^cD^q w(t) \geq f(t, w) + g(t, v), & \beta_0(0) \geq u_0. \end{cases}$$

See [11] for other types of coupled lower and upper solutions relative to (2.6). Denoting $F(t, u) = f(t, u) + g(t, u)$, we state the next comparison result.

Theorem 2.8. *Let v, w be natural lower and upper solutions of* (2.6), *respectively. Suppose that $F(t, u_1) - F(t, u_2) \leq L(u_1 - u_1)$ whenever $u_1 \geq u_2$, where L is a constant such that $L > 0$, then $v(0) \leq w(0)$ implies that $v(t) \leq w(t)$, $t \in J$.*

Proof. See [11, 18] for details. $\qquad\square$

Note that if v, w are coupled lower and upper solutions of type I for (2.6), then result of Theorem 2.8 holds true if the one sided Lipschitz condition of $F(t, u)$ is replaced by $f(t, u_1) - f(t, u_2) \leq L_1(u_1 - u_2)$ and $g(t, u_1) - g(t, u_2) \geq -L_2(u_1 - u_2)$ for $u_1 \geq u_2$, where L_1 and L_2, are constants greater than zero.

Theorem 2.9. *Suppose $v, w \in C^1[J, \mathbb{R}]$ are natural lower and upper solutions of type I of (2.6) such that $v(t) \leq w(t)$ on J and $F \in C(\Omega, \mathbb{R})$. Then there exists a solution $u(t)$ of (2.6) such that $v(t) \leq u(t) \leq w(t)$ on J, provided $v(0) \leq u_0 \leq w(0)$.*

Proof. See [6] for details. $\qquad\square$

Note that coupled lower and upper solutions of type I for (2.6), implies that they are natural lower solutions for (2.6), if $g(t, u)$ of (2.6) is non increasing in u for $t \in J$. In this situation, the conclusion of Theorem 2.9 is true with coupled lower and upper solutions of type I. The next theorem provides the generalized monotone method to obtain the coupled minimal and maxima solutions of (2.6).

Theorem 2.10. *Assume that*

(i) *$v_0, w_0 \in C^1[J, \mathbb{R}]$. v_0, w_0 are coupled lower and upper solutions of (2.6) of type I, with $v_0(t) \leq w_0(t)$ on J.*

(ii) *$f(t, u), g(t, u) \in C[J \times \mathbb{R}, \mathbb{R}]$, where $f(t, u)$ is non-decreasing in u on J, and $g(t, u)$ is non-increasing in u on J.*

Then there exist monotone sequences, $v_n(t)$ and $w_n(t)$, such that $v_n(t) \to v(t)$ and $w_n(t) \to w(t)$ uniformly and monotonically, where $v(t)$ and $w(t)$ are coupled minimal and maximal solutions of equation (2.6) on J. That is, for any solution $u(t)$ of (2.6) with $v_0 \le u \le w_0$ on J, we get natural sequences, $\{v_n\}$ and $\{w_n\}$, satisfying, $v_0(t) \le v_1(t) \le v_2(t) \le \cdots \le v_n(t) \le u(t) \le w_n(t) \le \cdots \le w_2(t) \le w_1(t) \le w_0(t)$, on J, where $v(t)$ and $w(t)$ satisfy the coupled system,

$$(2.7) \qquad \begin{aligned} {}^cD^q v(t) &= f(t,v(t)) + g(t,w(t)), \quad v(0) = u_0, \\ {}^cD^q w(t) &= f(t,w(t)) + g(t,v(t)), \quad w(0) = u_0. \end{aligned}$$

Here we use the following iterative schemes(namely type I iterative schemes),

$$\begin{aligned} {}^cD^q v_{n+1}(t) &= f(t,v_n(t)) + g(t,w_n(t)), \quad v_{n+1}(0) = u_0, \\ {}^cD^q w_{n+1}(t) &= f(t,w_n(t)) + g(t,v_n(t)), \quad v_{n+1}(0) = u_0. \end{aligned}$$

Also, $v(t) \le u(t) \le w(t)$ on J.

See [11] for details of the proof.

Remark 2.11. The conclusion of Theorem 2.10 holds true with natural lower and upper solutions of (2.6) provided with further assumptions that $v_0 \le v_1$ and $w_1 \le w_0$ for $t \in J$. However, in general $v_0 \le v_1$ and $w_1 \le w_0$ is true $t \in [0,\bar{t}]$ for $\bar{t} < T$. See numerical results in [14] for integer derivatives and [15, 16] for fractional derivatives for details.

3. MAIN RESULTS

In the preliminary results we saw that the generalized monotone method combined with coupled lower and upper solutions are very useful in computing the solution of Caputo fractional differential equations (2.6) when uniqueness conditions are satisfied. The advantage of the method is that each iterates are computed using (2.4). Natural lower and upper solutions are relatively easy to compute. For example, the equilibrium solutions are natural lower and upper solutions and it guarantees that global solution or solutions exists. The equilibrium solutions are constant solutions. However, computing coupled lower and upper solution are not trivial since constants will never be coupled and lower solutions. To demonstrate that let k, K be coupled lower and upper solutions of (2.6) such that $k \le K$. Then, it follows that $0 = {}^cD^q k \le f(t,k) + g(t,K) \le f(t,K) + g(t,k) \le {}^cD^q K = 0$, using the increasing and decreasing nature of $f(t,u)$ and $g(t,u)$ respectively, and the fact that $k \le K$. This was the motivation to compute the next result. In this paper, we just recall the result and for proof and other details including numerical applications see [15]. Also see [16] for the extension of these results to Caputo fractional differential systems and its application to Lotka-Volterra type of fractional differential equations.

Theorem 3.1. *Assume that*

(i) $v_0, w_0 \in C[J, \mathbb{R}]$ *are natural lower and upper solutions of (2.6) such that* $v_0(t) \leq$ *$w_0(t)$ on J.*

(ii) $f, g \in C[J \times \mathbb{R}, \mathbb{R}], f(t, u)$ *is nondecreasing and $g(t, u)$ is nonincreasing in u on J. Then there exists monotone sequences $\{v_n(t)\}$ and $\{w_n(t)\}$ on J such that $v_n(t) \to v(t)$ and $w_n(t) \to w(t)$ uniformly and monotonically to v and w where v and w are coupled lower and upper solutions of (2.6) such that $v \leq w$ on J. The iterative scheme is given by*

$$^cD^q v_{n+1}(t) = f(t, v_n) + g(t, w_n), \text{ on } [0, t_n], v_{n+1}(0) = u_0$$

$$^cD^q w_{n+1}(t) = f(t, w_n) + g(t, v_n), \text{ on } [0, \overline{t_n}], w_{n+1}(0) = u_0,$$

where $v_n(t) \geq v_0(t)$ on $[0, t_n)$ and $w_n(t) \leq w_0(t)$ on $[0, \overline{t_n})$.

Also define $v_{n+1}(t), w_{n+1}(t)$ on $[t_n, T]$ and $[\overline{t_n}, T]$ respectively as the solution of

$$^cD^q v_{n+1}(t) = f(t, v_n) + g(t, w_n), \quad v_{n+1}(t_n) = \lim_{h \to 0} v_{n+1}(t_n - h),$$

$$^cD^q w_{n+1}(t) = f(t, w_n) + g(t, v_n) \quad w_{n+1}(\overline{t_n}) = \lim_{h \to 0} w_{n+1}(\overline{t_n} - h).$$

The generalized monotone method provides linear convergence, which essentially means the convergence rate is slow. The next result accelerates the rate of convergence. See [15] for details of the proof.

Theorem 3.2. *Let all the hypothesis of Theorem 2.10 hold. Then there exist monotone sequences v_n and w_n, where the iterative scheme is given by*

(3.1)
$$^cD^q v_{n+1}^* = f(t, v_n^*) + g(t, w_n^*), \quad v_{n+1}^*(0) = u_0,$$
$$^cD^q w_{n+1}^* = f(t, w_n^*) + g(t, v_{n+1}^*), \quad w_{n+1}^*(0) = u_0,$$

where $v_0^* = v_1$ and w_0^* is the solution of $^cD^q w_0^* = f(t, w_0) + g(t, v_1), w_0^*(0) = u_0$, **or**

(3.2)
$$^cD^q v_{n+1}^* = f(t, v_n^*) + g(t, w_{n+1}^*), \quad v_{n+1}^*(0) = u_0,$$
$$^cD^q w_{n+1}^* = f(t, w_n^*) + g(t, v_n^*), \quad w_{n+1}^*(0) = u_0,$$

where $w_0^* = w_1$ and v_0^* is the solution of $^cD^q v_0^* = f(t, v_0) + g(t, w_1), v_0^*(0) = u_0$.

4. CONCLUSION

Computing the solution of nonlinear ordinary Caputo differential equation is a challenge since it does not enjoy the nice properties of the integer derivative like the product rule for the derivatives. Generalized monotone method combined with coupled lower and upper solutions proves to be an efficient and fruitful tool to compute the coupled minimal and maximal solutions. If uniqueness conditions are satisfied the coupled minimal and maximal solutions solutions will converge to the unique solution of the nonlinear problem. In order to use generalized monotone method the challenge is to compute the coupled lower and upper solution to the desired interval or to the interval of existence established by upper and lower solutions. Using the

scheme similar to generalized monotone method Theorem 3.1 provides a method to compute the coupled lower and upper solutions to any desired interval. The rate of convergence can be slightly accelerated by using the scheme of Theorem 3.2. In [17] a mixed generalized iterative method to compute the solution of the nonlinear problem (2.6) has been developed when $f(t, u)$ is convex in u for $t \in J$ and $g(t, u)$, and decreasing in u for $t \in J$. In addition, the method yields super linear convergence. The method is both theoretical and computational. Computational methods have the following challenges and lead to open problems. These challenges are partly due to the fact that some of the nice properties enjoyed by the exponential function does not hold good for Mittag-Leffler function. In [9] is monograph which is completely dedicated to Mittag-Leffler function and its application. However the exponential properties of Mittag-Leffler function is still open. Here, we mention a few of them.

(i) Each iterates are two decoupled systems with variable coefficients. There is no closed form of solution for the linear Caputo fractional differential equations with variable coefficients. See [13] for recent result on linear fractional differential equation with variable coefficent;

(ii) In the simple situation when $f(t, u)$ is linear, if we use the generalized monotone iterative method we need to solve the linear equation with constant coefficients. The first step is simple since we can use (2.4). The second step involves the use of (2.4) where $f(t)$ is either a Mittag-Leffler function or the product of Mittag-Leffler functions. This result is yet to be established so that the solution part related to the nonhomogeneous part can be computed accurately. The result for $q = 1$ is very trivial, that is $e^{\lambda t} \times e^{\mu t} = e^{\lambda + \mu t}$.

(ii) Theoretical extension of generalized monotone method has been established in [18] for Caputo fractional nonlinear reaction diffusion equation. The computation of a simple linear Caputo fractional reaction diffusion requires us to prove the convergence of an infinite series whose elements are Mittag-Leffler functions. This requires us to establish the exponential properties of the Mittag-Leffler function, which is a well known trivial result for the exponential function. That is, $e^{\lambda t} \times e^{-\lambda t} = 1$. This relation is not true for Mittag-Leffler function.

REFERENCES

[1] M. Caputo, Linear models of dissipation whose Q is almost independent, II, Geophysical Journal of the Royal Astronomical Society, **13** (1967), 529–539.

[2] Z. Denton, P. W. Ng, and A. S. Vatsala, Quasilinearization Method Via Lower and Upper Solutions for Riemann-Liouville Fractional Differential Equations "Nonlinear Dynamics and Systems Theory, 11 (3) (2011), 239–251.

[3] Kai Diethelm, *The Analysis of Fractional Differential Equations*, Springer, 2004.

[4] A. A. Kilbas, H. M. Srivatsava and J. J. Trujillo, *Theory and Applications of Fractional Differential Equations*, Elsevier, Amsterdam, 2006.

[5] V. Lakshmikantham, S. Leela,*Differential and Integral Inequalities*Volume I, II, Academic Press, New York, 1969.

[6] V. Lakshmikantham, S. Leela, and D. J. Vasundhara Devi, *Theory of fractional dynamic systems*, Cambridge Scientific Publishers, 2009.

[7] G. S. Ladde, V. Lakshmikantham and A. S. Vatsala, *Monotone Iterative Techniques for Nonlinear Differential Equations*, Pitman, (1985).

[8] V. Lakshmikantham and A. S. Vatsala, *Generalized Quasilinearization for Nonlinear Problems*, Kluwer Academic Publishers,Dordrecht,Boston,London,

[9] R. Gorenflo, A. A. Kilbas, F. Mainardi, S. V. Rogosin, *Mittag-Leffler Functions, Related Topics and Applications*, Springer: Berlin, Germany, 2014; p. 443.

[10] B. Oldham and J. Spanier, *The Fractional Calculus*, Academic Press, New York, London, 1974.

[11] TH. T Pham, J. D. Ramírez, A. S. Vatsala, Generalized Monotone Method for Caputo Fractional Differential Equation with Applications to Population Models, Neural,Parallel ans Scientific Computations 20 (2012) 119–132.

[12] I. Podlubny, *Fractional Differential Equations.* Academics Press, San Diego, 1999.

[13] B. Sambandham and A. S. Vatsala, *Basic results for Sequential Caputo fractional differential equations* "Mathematics, (2015).

[14] M. Sowmya, A. S. Vatsala, Noel C, Sheila H, Zenia N, Dayonna P, and, Jasmine W, Numerical Application Of Generalized Monotone Method for Population Models, Neural, Parallel, and Scientific Computations, 20 359–372(2012).

[15] M. Sowmya and A. S. Vatsala, Numerical Approach via Generalized Monotone Method for Scalar Caputo Fractional Differential Equations, Neural, Parallel, and Scientific Computations 21 (2013) 19–30.

[16] M. Sowmya and A. S. Vatsala, Numerical Approach via Generalized Monotone Method for System of Caputo Fractional Differential Equations, Neural, Parallel, and Scientific Computations, 21 347–360 (2013).

[17] M. Sowmya and A. S. Vatsala,Superlinear Convergence For Caputo Fractional Differential Equations With Application, Dynamic Systems and Applications 22, 479–492, (2013).

[18] D.Stutson and A.S. Vatsala, Generalized Monotone Method for Caputo Fractional Differential Systems Via Coupled Lower and Upper Solutions, (2011).